Math 425A Problem Sheet 4 (due 9am on Monday, 27th Sep)

Essential problems

1. (3 pts) Let $(q_n) \subset \mathbb{R}$ be a sequence of all rational numbers (note it exists by Ex. 2.20.3), and let $x \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Show that there exists a subsequence $(q_{n_k})_{k\geq 1}$ such that $q_{n_k} \to x$ as $k \to \infty$. *(Hint: Use PS2.2(a).)*

2. (4 pts) In this exercise we give a proof of compactness of a closed interval $[a, b] \subset \mathbb{R}$ (Thm. 4.5) that is somewhat similar to the proof from class, but does not use the notion of sequential compactness. Let $\{A_\alpha\}_{\alpha\in I}$ be an open cover of $[a, b]$. Let

$$C := \{c \in [a, b] \colon \text{ there exists a finite subcover that covers } [a, c]\}$$

(a) Show that C is nonempty and bounded above. Deduce that $x := \sup C$ exists and $x \in [a, b]$.
(b) Show that there exists $\alpha_x \in I$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A_{\alpha_x}$.
(c) Use the definition of supremum and part (b) to show that $x \in C$.
(d) Use an argument by contradiction to deduce that $x = b$, and hence conclude that $[a, b]$ is compact (using Definition 3.14).

3. (3 pts) Let

$$d(x, y) := |2^{-x} - 2^{-y}| \qquad \text{for } x, y \in \mathbb{R}.$$

Then d is a metric on $\mathbb{R}$. Show that $(\mathbb{R}, d)$ is not a complete metric space. *(Hint: Consider the sequence $(x_n) := (n)$, and use the fact that $2^{-n} \to 0$ as $n \to \infty$.)*
(Comment: This gives an example of a metric on $\mathbb{R}$ with respect to which $\mathbb{R}$ is not complete. (Recall that $\mathbb{R}$ is complete with respect to the (usual) Euclidean metric (by Lem. 4.20).))

Additional problems

4. (1 pt) Let (X, d) be a complete metric space, and let $Y \subset X$ be any subset. Show that (Y, d) is complete if and only if Y is a closed subset of X.

5. (1 pt) Let (X, d) be a compact metric space (i.e. X is compact). Show that (X, d) is complete. *(Hint: use the proof of Lem. 4.20.)*
Give an example of a complete metric space that is not compact.

6. (1 pt) Let (X, d) be a metric space, and $Y \subset X$. Show that if Y is totally bounded (i.e. that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $Y \subset \cup_{k=1}^{n_\varepsilon} N_\varepsilon(x_k)$ for some $x_1, \ldots, x_{n_\varepsilon} \in X$, recall (3.4)), then it is bounded.

7. (1 pt) Let X be any infinite set and

$$d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

(This is the *discrete metric* on X.) Show that (X, d) is a complete metric space. Show that X is not compact.

8. (1 pt) Let (X, d) be as in question 7. Show that X is bounded, but not totally bounded. *(Comment: this shows that the implication opposite to the one in question 6 is false.)*

9. (1 pt) Let (X, d) be a metric space. Show that if $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset X$ are Cauchy, then the sequence of distances, $(d(x_n, y_n))_{n\geq 1} \subset \mathbb{R}$, converges in $\mathbb{R}$. Furthermore, show that for each $m \in \mathbb{N}$ the sequence $(d(x_n, y_{n+m}))_{n\geq 1}$ converges to the same limit. *(Hint: Use the trick from the proof of Ex. 4.13 and apply the triangle inequality.)*

10. (1 pt) Let $(X, d) := ((-1, 1), |\cdot|)$ and $a_n := (-1)^n \frac{n-1}{n} \in X$ for $n \in \mathbb{N}$. Show that (a_n) is not Cauchy in two ways: first using the definition (Def. 4.15), and then using completeness of $\mathbb{R}$ (i.e. using Lem. 4.20).

11. (1 pt) Show that if $(x_n) \subset \mathbb{R}$ is such that $x_n \to \infty$ then $\sup\{x_n\} = \infty$, but that the opposite implication is false (i.e. give an example of a sequence $(x_n) \subset \mathbb{R}$ such that $\sup\{x_n\} = \infty$ but $x_n \not\to \infty$ as $n \to \infty$). *(Comment: this verifies the comment at Def. 4.21; note that $\{x_n\} := \{x_n : n \in \mathbb{N}\}$ denotes the* ***set*** *of all elements of the* ***sequence*** *(x_n).)*

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