MATH 425a PS 4

1.

We take it for granted that $\mathbb{Q}$ is dense in $\mathbb{R}$.

That is, ~~for $x\in\mathbb{R}$~~, $\forall x \in \mathbb{R}, \varepsilon>0, \exists q\in\mathbb{Q}$ s.t. $|q-x|<\varepsilon$. $(q\in(x-\varepsilon, x+\varepsilon).)$

Thus, for $n\in\mathbb{N}$, $a-\frac{1}{n}<a+\frac{1}{n}$, $\exists q_n$ s.t. $|q_n-x|<\frac{1}{n}$

↳ let $\varepsilon>0$, $\varepsilon=\frac{1}{n}$.

~~Thus q_n converges to $x \Rightarrow q_{n_k}\to x$ as $k\to\infty$.~~

Construct $\{q_{n_k}\}$ s.t. for q_{n_1} be any $q\in\mathbb{Q}$ s.t. $|q_{n_1}-x|<1$.

let $n_2>n_1$. $q_{n_2}\in\mathbb{Q}$ s.t. $|q_{n_2}-x|<\frac{1}{2}$. ↑ We can always find a qualified n_2 because there are infinite rational numbers in $(x-\frac{1}{2}, x+\frac{1}{2})$ but only finite points before n_1.

Thus, for $n_k>\cdots>n_2>n_1$, let $q_{n_k}\in\mathbb{Q}$ s.t. $|q_{n_k}-x|<\frac{1}{k}=\varepsilon$.
$(k\in\mathbb{N}), \varepsilon>0$

This sequence $\{q_n\}$ we just constructed will converge to x. Thus $q_{n_k}\to x$ as $k\to\infty$

2.

a) C is nonempty because it contains a, b. C is bounded above by b by definition.
$\because \mathbb{R}$ has LUBP property, $\therefore x:=\sup C$ exists. $x\in[a,b]$.

b) $\because x\in[a,b]$, $\therefore x$ is covered by some random open cover A_{α_x}.
↳ $\exists\varepsilon>0$ s.t. $N_\varepsilon(x)\subset A_{\alpha_x}$ by definition of openess. $\Rightarrow (x-\varepsilon, x+\varepsilon)\subset A_{\alpha_x}$.
$\alpha_x\in I$

c) $\forall\varepsilon>0$, $x-\frac{\varepsilon}{2}$ must lie within C $(x:=\sup C)$.
Suppose $\{A_{\alpha_1}, A_{\alpha_2}\cdots A_{\alpha_n}\}$ covers $[a, x-\frac{\varepsilon}{2}]$. ~~This is~~ There are finitely many subcovers.
Thus $\{A_{\alpha_1}, A_{\alpha_2}\cdots A_{\alpha_n}, A_{\alpha_x}\}$ covers $[a,x]$. (result from b)).
↳ Finitely many subcovers as well. Thus $x\in C$.

d) We know from c) that $[a,x]\subset\bigcup_{i=1}^{n}A_{\alpha_i}\cup A_{\alpha_x}$.
However, $x+\frac{\varepsilon}{2}\in A_{\alpha_x}$. If $x\neq b$, then $[a, \min(x+\frac{\varepsilon}{2}, b)]\subset\bigcup_{i=1}^{n}A_{\alpha_i}\cup A_{\alpha_x}\subset C$.
$\because x+\frac{\varepsilon}{2}, b$ would be greater than x in this case, $\therefore$ ~~sup C~~ $x\neq\sup C$.
Since there's a contradiction, $x=b$.

3.

$d(x,y) = \left|\frac{1}{2^x} - \frac{1}{2^y}\right|$ for $x,y \in R$.

Construct $\{a_n\}$ s.t. $\{a_n\} = n$.

$d(a_n, a_m) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| \le \frac{1}{2^n} + \frac{1}{2^m}$.

$\hookrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t. $d(a_n, a_m) \le \frac{1}{2^n} + \frac{1}{2^m} < \varepsilon$ when $n, m > N$.

$\therefore \{a_n\}$ is a Cauchy sequence.

Suppose $\{a_n\}$ converge to $x \in R$. $d(2^{-n}, 2^{-x}) < \varepsilon$ for $\varepsilon > 0$, $n \ge N \in \mathbb{N}$.

Suppose $\varepsilon < 2^{-(x+1)}$. let n be large s.t. $2^{-n} < \frac{2^{-x}}{3}$. This is reasonable.

Then $d(x, x_n) = \frac{1}{2^x} - \frac{1}{2^n} > (1 - \frac{1}{3})\frac{1}{2^x} > \frac{1}{2}\frac{1}{2^x} = \frac{1}{2^{x+1}} = \varepsilon$.

Thus ~~$d(x, x_n) \not\to$~~ $\{a_n\}$ fails to converge in R.

There is a contradiction, and (X, d) is thus not complete.

4.

$\Rightarrow$: let $\{x_n\}$ be a Cauchy sequence in ~~X~~ Y. Since X is complete, $x_n \to a \in X$.

But Y is closed, so Y contains all its limit points. So ~~$x \in X$~~ $x \in Y$.

Hence ~~X Y is com~~ (Y, d) is complete.

$\Leftarrow$: If (Y, d) complete, $\exists$ ~~$\{x_n\}$~~ $\{y_n\} \subset Y$ converging to a. ~~Th~~ Obviously this is a Cauchy sequence, and $a \in Y$. However, the limit point of a sequence is unique in a metric space. Thus Y contains all its limit points, meaning it's a closed subset of X.

5.

let $\{x_n\}$ be a Cauchy sequence in X.

$\because X$ compact $\therefore \exists$ convergent subsequence $\{x_{n_k}\} \to x \in X$.

We want to show $|x_n - x| < \varepsilon$ for $\varepsilon > 0$.

let N_1 be: when $n_k \ge N_1$, $|x_{n_k} - x| < \frac{\varepsilon}{2}$.

N_2 ...: when $m, n \ge N_2$, $|x_n - x_m| < \frac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$.

$|x_n - x| \le |x_n - x_N| + |x_N - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow$ converges in X.

Thus X is complete.

7.

(X, d) is complete because only trivial sequences count as Cauchy sequences here. $(x=y)$
of course $\{x_n, y_n\}$ converges to 0 (otherwise, distance between 2 diff. points always $=1$.
↳ not Cauchy.

Not compact: We know $\{x\}$ an open cover for X. ~~let A be a infinite subset of X:~~
↳ $X \subset \bigcup_{x \in X} \{x\}$. clearly there are infinite elements in X.
Thus, $\nexists\, k$ s.t. $X \subset \bigcup_{i=1}^{k} \{x_i\}$. #
In other words, we can't extract a finite subcover that covers X.
Hence X is not compact.