Solutions to Problem Set 4

Math 425a, Fall 2021

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1. Let $\{q_n\}_{n\in\mathbb{N}}$ be the given enumeration of rationals. Given any $x \in \mathbb{R}$, we know from the density of rationals in $\mathbb{R}$ that there exists a sequence of rationals converging to x. However, that is not quite enough because the terms in this sequence can appear in any order and therefore we cannot guarantee that it will be a subsequence of the given enumeration. So we need to construct our sequence a bit more carefully.

   First choose p_1 to be any rational in the interval $(x-1, x+1)$. Let the index number of p_1 in the given sequence of rationals be n_1, i.e. $p_1 = q_{n_1}$. Next we consider the interval $(x - \frac{1}{2}, x + \frac{1}{2})$, and we choose p_2 from here such that the index number of p_2 in the given sequence is larger than n_1 (this can always be ensured because there are infinitely many rationals in the interval*and there are only finitely many rationals whose index numbers are less than n_1). In other words we choose $p_2 \in \mathbb{Q}$ such that $p_2 = q_{n_2}$ for some $n_2 > n_1$. Proceeding this way, in the k^{th}-step we look at the interval $(x - \frac{1}{k}, x + \frac{1}{k})$ and choose a rational whose index number is bigger that n_{k-1} i.e, the index of the previously chosen rational q_{k-1}. This way we obtain q_{n_k}.

   We now claim that this sequence $\{q_k\}$ that we have constructed is the desired subsequence. Indeed, as for all k we have $|q_{n_k} - x| < \frac{1}{k}$ by construction, the sequence does converge to x and their index numbers in the original enumeration satisfy the condition that $n_1 < n_2 < \ldots < n_k < \cdots$, therefore making it a subsequence too.

2. (a) Let $\{A_\alpha\}_{\alpha\in I}$ and C be as given. Clearly C is bounded above by b because it is a subset of $[a,b]$. C is nonempty because the interval $[a,a]$ is just $\{a\}$ and it has a finite subcover: just choose any one A_α that contains a (and so obtain a 1-element cover, in particular finite). Such a choice exists because, by assumption, the collection $\{A_\alpha\}_{\alpha\in I}$ covers the whole interval $[a,b]$. Therefore C is bounded above and nonempty and hence $x := \sup C$ (by the least upper bound property, Thm. 1.16). Since b is another upper bound we can conclude that $x \leq b$ and since $a \in C$ we have $a \leq x$. Combining we get that $x \in [a,b]$.

   (b) From part (a) we have that $x \in [a,b]$. As $\{A_\alpha\}_{\alpha\in I}$ covers the whole of $[a,b]$ we can find A_{α_x} that contains x. By assumption, A_{α_x} is open, so we can find $\varepsilon_x > 0$

*This is a minor sharpening of the density property of $\mathbb{Q}$ in $\mathbb{R}$, PS2.2(a): recall that for any $a < b$ in $\mathbb{R}$ we can find $q_0 \in \mathbb{Q}$ such that $a < q_0 < b$. Now carrying on the same argument, we can find $q_1 \in \mathbb{Q}$ such that $a < q_1 < q_0$ and so on.

such that $(x - \varepsilon_x, x + \varepsilon_x) \subset A_{\alpha_x}$.

(c) Since x is the supremum of C we have that $x - \varepsilon_x$ is not an upper bound of C and so we can choose $c \in C$ such that $c > x - \varepsilon_x$. However as $c \in C$ we have a finite subcover of the interval $[a, c]$. Let this finite subcover consist of the sets $\{A_1, \cdots, A_k\}$. Using part (b) then we have-

$$[a, x] \subset [a, x + \varepsilon_x) = [a, c] \cup (x - \varepsilon_x, x + \varepsilon_x) \subset \underbrace{\left(\bigcup_{n=1}^{n=k} A_n \right) \cup A_{\alpha_x}}_{(k+1) \text{ sets}}$$

proving that $[a, x]$ is covered by finitely many elements of the collection $\{A_\alpha\}_{\alpha \in I}$. Therefore $x \in C$.

(d) Let us assume by way of contradiction that $x < b$. Choose ε' such that $x + \varepsilon' < b$ and $\varepsilon' < \varepsilon_x$. But then using part (c),

$$[a, x + \varepsilon'] \subset [a, x + \varepsilon_x) \subset \underbrace{\left(\bigcup_{n=1}^{n=k} A_n \right) \cup A_{\alpha_x}}_{\text{finite collection}}$$

which implies $x + \varepsilon'$ is in C, contradicting the fact that x is the supremum. Therefore, we must have $x \geq b$. However as b was an upper bound of C as we observed in part (a), we must also have $x \leq b$. So we can conclude that $x = b$.

3. Following the hint we consider the sequence $\{x_n\} = \{n\}$ and show that it is Cauchy with respect to the new metric. Indeed for $m > n$,

$$d(x_n, x_m) = \left| \frac{1}{2^n} - \frac{1}{2^m} \right| = \frac{1}{2^n} \left| 1 - \frac{1}{2^{m-n}} \right| \leq \frac{1}{2^n} \left(1 + \frac{1}{2^{m-n}} \right)$$

Since $m > n$ we have $m - n \geq 1$ which implies that $\frac{1}{2^{m-n}} \leq \frac{1}{2}$ and so $d(x_n, x_m) \leq \frac{3}{2} \frac{1}{2^n}$. Therefore for any $\varepsilon > 0$ if we choose $m > n > \log_2(\frac{3}{2\varepsilon})$ we have $d(x_n, x_m) < \varepsilon$. This proves that the sequence is Cauchy. However, this sequence does not have a limit in $\mathbb{R}$L Suppose otherwise that there exists $x \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(n, x) < \varepsilon \qquad \text{for } n \geq N.$$

Then

$$|2^{-x}| \leq |2^{-n}| + |2^{-n} + 2^{-x}| = |2^{-n}| + d(n, x) \leq 2\varepsilon$$

for $n \geq N$ and such that $2^{-n} < \varepsilon/2$. Since the left-hand side does not depend on ε we now see that we must have $2^{-x} = 0$, which is a contradiction (there is no such $x \in \mathbb{R}$).

4. Recall Def. 4.21 that a sequence $\{x_n\}$ is said to converge to infinity if given any $M > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $x_n > M$. This however implies that no $M > 0$ is an upper bound for the sequence which implies that $\{x_n\}$ is an unbounded set, i.e. $\sup\{x_n\} = \infty$.

To show that the opposite implication is false, consider the following sequence

$$x_n = \begin{cases} n & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

Since infinitely many terms in this sequence equal 1, if we choose $M = 2$, there is no $N \in \mathbb{N}$ such that $x_n > 2$ for all $n > N$. So the sequence does not converge to ∞. On the other hand, clearly $\sup\{x_n\} = \infty$.

5. First assume Y is closed. Consider a Cauchy sequence $\{y_n\} \in Y$. Since all Cauchy sequences are convergent in X (since (X, d) is complete, recall Def. 4.18) we know that there exists some $y \in X$ such that $y_n \to y$. However that makes y a limit point of Y. By assumption Y is closed so it contains all its limit points implying $y \in Y$. So we have proved any Cauchy sequence in Y converges to a point inside Y concluding that (Y, d) is complete.

   On the other hand, suppose that (Y, d) is a complete metric space, and take any sequence $(y_n) \subset Y$ such that $y_n \to x$ for some $x \in X$. We need to show that $x \in Y$ (i.e. then Y is closed). Since all convergent sequences are Cauchy (Lem. 4.17) (y_n) is a Cauchy sequence, and so (since (Y, d) is complete) there exists $y \in Y$ such that $y_n \to y$. By uniqueness of limits (Lem. 3.5) we must have $x = y$, and so $x \in Y$, as required.

6. Let $\{x_n\}$ be a Cauchy sequence in X. Since X is compact there exists a convergent subsequence x_{n_k} converging to some point $x \in X$ (by sequential compactness, Thm. 3.16). We'll prove that this forces the whole sequence to converge to x.

   Note that $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$. Fix some $\varepsilon > 0$, Since $\{x_n\}$ is Cauchy we can find N_1 such that for $n, n_k > N_1$, $d(x_n, x_{n_k}) < \frac{\varepsilon}{2}$ and since $x_{n_k} \to x$ there exists N_2 such that for $n_k > N_2$ we have $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Therefore for $n > \max\{N_1, N_2\}$, we have $d(x_n, x) < \varepsilon$ proving that $x_n \to x$.

   Putting everything together, we have proved that any Cauchy sequence $\{x_n\} \in X$ converges to some point $x \in X$ which implies (X, d) is complete.

   For a counterexample to the opposite implication: $(\mathbb{R}, |.|)$ is complete but not compact.

7. Recall that a subset Y in (X, d) is bounded if there exists some $x \in X$ and $R > 0$ such that $Y \subset B(x, R)$. Suppose that Y is totally bounded and fix $\varepsilon := 1$. Then from the given condition there exists $n_1 \in \mathbb{N}$ and points $x_1, x_2, \cdots, x_{n_1}$ such that

   $$Y \subset \bigcup_{k=1}^{k=n_1} B(x_k, 1) \tag{1}$$

   Choose $R = 1 + \max_{1 \leq k \leq n_1} d(x_1, x_k)$.

   Now note that if $y \in B(x_k, 1)$ for some k then $d(y, x_1) \leq d(y, x_k) + d(x_k, x_1) < 1 + d(x_k, x_1) < R$. Therefore we have that $B(x_k, 1) \subset B(x_1, R)$ for all $1 \leq k \leq n_1$.

Taking unions this implies

$$\bigcup_{k=1}^{k=n_1} B(x_k, 1) \subset B(x_1, R)$$

Combining this with (1), gives us $Y \subset B(x_1, R)$ implying Y is bounded.

8. Observe that by the definition of the metric we have $d(x, y) < 1 \iff x = y$. Now consider a Cauchy sequence $\{x_n\} \in X$. Then there exists N such that for $m.n > N$ we have $d(x_m, x_n) < 1$. Therefore in this special case this amounts to the same thing as $x_m = x_n$ for all $m, n > N$, which obviously implies that this sequence converges as all terms become equal after some point.

   Note that by the special metric that we have all balls of radius less than 1 consist just of their center. Now write X as follows-

$$X = \bigcup_{x \in X} \{x\} = \bigcup_{x \in X} B(x, \frac{1}{2})$$

   This is an open cover of X and since X is infinite it consists of infinitely many sets. Clearly this open cover has no finite subcover because the balls are disjoint and hence the center of each ball is not contained in any other. Hence X is not compact.

9. Let $\varepsilon := \frac{1}{2}$. If X were totally bounded then we would have had some $n_\varepsilon \in \mathbb{N}$ and $x_1, x_2, \cdots, x_{n_\varepsilon} \in X$ such that

$$X \subset \bigcup_{k=1}^{k=n_\varepsilon} B(x_k, \varepsilon) \tag{2}$$

   However, because of the metric, as we have observed before that $B(x_k, \varepsilon) = B(x_k, \frac{1}{2}) = \{x_k\}$. From (2) then we have

$$X \subset \bigcup_{k=1}^{k=n_\varepsilon} \{x_k\}$$

   However, X was assumed to be infinite and hence cannot be contained in any finite set leading to a contradiction.

10. Let $z_n := d(x_n, y_n)$. We'll show $\{z_n\}$ is a Cauchy sequence in $(\mathbb{R},|.|)$ and hence by the completeness of $(\mathbb{R},|.|)$ (Lem. 4.20) we can conclude that it converges. To that end note

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

   which implies

$$z_n - z_m \leq d(x_n, x_m) + d(y_m, y_n) \tag{3}$$

   Similarly,

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$$

   which gives us

$$z_n - z_m \geq -d(x_n, x_m) - d(y_m, y_n) \tag{4}$$

Combining (3) and (4) yields,

$$|z_n - z_m| \leq d(x_n, x_m) + d(y_m, y_n)$$

Now fix $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy there exists N_1 such that for all $m, n > N_1$ we have $d(x_m, x_n) < \frac{\varepsilon}{2}$ and since $\{y_n\}$ is Cauchy there exists N_2 such that for all $m, n > N_2$ we have $d(y_m, y_n) < \frac{\varepsilon}{2}$. Thus for any $m, n > \max\{N_1, N_2\}$ we have $|z_n - z_m| \leq d(x_n, x_m) + d(y_m, y_n) < \varepsilon$. Therefore $\{z_n\}$ is Cauchy as desired.

As for the second claim fix $m \geq 0$ and $\varepsilon > 0$. Note that the triangle inequality gives

$$d(x_n, y_{n+m}) \leq z_n + d(y_n, y_{n+m})$$

as well as

$$z_n \leq d(y_n, y_{n+m}) + d(x_n, y_{n+m}).$$

Together these two inequalities give

$$|d(x_n, y_{n+m}) - z_n| \leq d(y_n, y_{n+m})$$

and the right hand side is bounded by $\varepsilon/2$ for all $n \geq N$, where N is a sufficiently large natural number (since (y_n) is Cauchy). Thus, letting $z \in \mathbb{R}$ be the limit of (z_n) we have

$$|d(x_n, y_{n+m}) - z| \leq |d(x_n, y_{n+m}) - z_n| + |z_n - z| \leq \varepsilon$$

if we take N sufficiently large so that also $|z_n - z| \leq \varepsilon/2$ for $n \geq N$.

11. (a) We'll show $\{a_n\}$ is not Cauchy from definition. Recall that the negation of the Cauchy condition is that there exists some $\varepsilon > 0$ such that for all N we can find indices $m, n > N$ satisfying $|a_m - a_n| > \varepsilon$. Fix some $N > 0$, choose an index $m > N$ such that m is even. Set $n = m + 1$. Then

$$|x_m - x_n| = |x_m - x_{m+1}| = \left| \frac{m-1}{m} - (-1)^{m+1} \frac{(m+1)-1}{m+1} \right| = \frac{m-1}{m} + \frac{m}{m+1}$$

where the last equality is because $(m+1)$ is odd.

Now note that $\frac{m-1}{m} + \frac{m}{m+1} = 2 - \frac{1}{m} - \frac{1}{m+1}$. Since $m \geq 1$ we have $\frac{1}{m} \leq 1$ and $\frac{1}{m+1} \leq \frac{1}{2}$. Therefore $2 - \frac{1}{m} - \frac{1}{m+1} \geq \frac{1}{2}$ or in other words $|x_m - x_n| \geq \frac{1}{2}$. Now choose some positive $\varepsilon < \frac{1}{2}$. By the above calculations we have shown that for any $N > 0$ we have indices $m, n > N$ such that

$$|x_m - x_n| \geq \frac{1}{2} > \varepsilon$$

thereby negating the Cauchy condition.

(b) We shall show that $\{a_n\}$ is not Cauchy using the completeness of $(\mathbb{R},|.|)$. Note that if $\{a_n\}$ is Cauchy in $((-1,1),|.|)$ then it is also Cauchy in $(\mathbb{R},|.|)$ and hence convergent in $\mathbb{R}$ (here we are using Lem. 4.20 that $(\mathbb{R},|.|)$ is complete). But if we look at the subsequence of even terms it converges to $+1$ and the subsequence of odd terms converges to -1. This cannot happen in a convergent sequence because all subsequences of a convergent in $\mathbb{R}$ sequence converge to the same limit. So $\{a_n\}$ is not convergent and hence not Cauchy in $(\mathbb{R},|.|)$ and therefore also not Cauchy in $((-1,1),|.|)$.