Math 425A Problem Sheet 5 (due 9am on Monday, 4th Oct)

Essential problems

1. (3 pts) Let $a_0 := 2021$ and

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{1}{a_n}\right).$$

Show that a_n converges (in $\mathbb{R}$) and find the limit.

2. (3 pts) Let $q \in \mathbb{Q}$. Find the set of limit points (recall (5.2)) of the sequence $(x_n) \subset \mathbb{R}$, where

$$x_n := nq - [nq].$$

(Here $[y] := \sup\{m \in \mathbb{N} \colon m \leq y\}$ denotes the "floor function".) Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$. *(Hint: Use the division algorithm, recall the solution to Ex. 5.12, for example.)*

3. (4 pts) Let $(x_n), (y_n) \subset \mathbb{R}$ be sequences bounded below.

(a) Show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) Give an example of a pair of sequences (x_n), (y_n) for which the inequality in (a) is sharp (i.e. it holds with "$<$").

(c) Show that if (x_n) converges (i.e. $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$) then the inequality in (a) becomes an equality, i.e. $\limsup_{n\to\infty}(x_n + y_n) = x + \limsup_{n\to\infty} y_n$.

(d) State and prove an analogous (to (a)) claim about the lower limit. *(Hint: you may use question 7 below.)*

Additional problems

4. (1 pt) Prove the analogue of Ex. 4.13 and the squeeze theorem (Cor. 5.9) for infinite limits in $\mathbb{R}$, i.e. show that:

(a) If $x_n \to x \in \mathbb{R}$ and $y_n \to +\infty$ then $x_n + y_n \to +\infty$.

(b) If $x_n \to -\infty$ and $y_n \to +\infty$ then $x_n y_n \to -\infty$. *(Comment: note that this is what we (almost) used in the solution to Ex. 5.2.)*

(c) If $x_n \to -\infty$ and $m \in \mathbb{N}$ then $x_n^m \to +\infty$ (if m is even) or $x_n^m \to -\infty$ (if m is odd),

(d) If $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $a_n \to \infty$, then $b_n \to \infty$.

5. (1 pt) Let $x_1 := 1$ and $x_{n+1} := 2\left(2 - \frac{5}{x_n+3}\right)$ for $n \geq 1$.

(a) Use induction to prove that $x_n \in (0,2)$ for all $n \geq 1$.

(b) Deduce that x_n is nondecreasing.

(c) Use Theorem 5.1 to show that x_n converges, and find the limit. *(Hint: use the conclusion of question 8 below.)*

6. (1 pt) Find the set of limit points (recall (5.2)), $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$ of the following sequences

(a) $a_n := \frac{2n^2}{7} - \left[\frac{2n^2}{7}\right]$,

(b) $b_n := \frac{1}{2}\left(n-2-3\left[\frac{n-1}{3}\right]\right)\left(n-3-3\left[\frac{n-1}{3}\right]\right)$,
(c) $c_n := \left(-1-\frac{1}{n}\right)^n + \sin\frac{n\pi}{4}$.

7. (1 pt) Show that

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

8. (1 pt) Let (x_n) be a sequence of positive real numbers. Show that

$$\liminf_{n\to\infty}\frac{1}{x_n} = \frac{1}{\limsup_{n\to\infty} x_n}.$$

(Here we use the convention that $\frac{1}{+\infty} = 0^+$ (i.e. 0 approached by nonnegative numbers) and that $\frac{1}{0^+} = +\infty$.) Deduce that if $x_n \to x$ for some $x \in \mathbb{R}\setminus\{0\}$ then $\frac{1}{x_n} \to \frac{1}{x}$.

9. (1 pt) Let $(x_n) \subset (0,\infty)$ be a sequence such that $x_n \to \infty$. Prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = \mathrm{e},$$

where e is the Euler number (recall (5.1)). *(Hint: If (x_n) is a sequence of integers apply Ex. 5.3. If not consider the sequence $([x_n])$, and use some inequalities (you can use the fact that exponential functions are monotonic (i.e. $a^x \le a^y$ for $a>0$, $x<y$)) to apply the squeeze theorem (Cor. 5.9).)*

10. (1 pt) Let $m \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-n} = \mathrm{e}^m.$$

(Hint: Use question 9 and Ex. 4.13.4.)

11. (1 pt) Show that $\sqrt[n]{n} \to 1$ as $n\to\infty$. *(Hint: Try a similar idea as in Ex. 5.10.)*

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