MATH 425a PS 5

1.

We clearly know $\{a_n\} > 0$ because a_n, $\frac{1}{a_n}$ are non-negative.
we also know $a_n + \frac{1}{a_n} > 2$ $\forall n$. To prove this, let $a_n + \frac{1}{a_n} < 2$.
$\hookrightarrow \frac{a_n^2+1}{a_n} < 2 \Rightarrow a_n^2 - 2a_n + 1 < 0$. $(a_n - 1)^2 < 0$. Clearly a contradiction.
Finally we know $\{a_n\}$ is monotonically decreasing.
$\hookrightarrow$ This is because $a_{n+1} - a_n = \frac{1}{2a_n} - \frac{1}{2}a_n = \frac{1}{2}(\frac{1}{a_n} - a_n)$
we know $a_n > 1 = \frac{1}{2} \cdot 2$. Therefore, $a_{n+1} < a_n \Rightarrow \{a_n\}$ has a lower bound 1.
$\{a_n\}$ thus converges. We know $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = x$. ~~The~~
$\hookrightarrow$ Then $x = \frac{1}{2}(x + \frac{1}{x})$. $x - \frac{1}{x} = 0 \Rightarrow x = 1$ or -1 (rejected since $x > 0$).
$\therefore \{a_n\}$ converges to 1.

2. We know $0 \leq \liminf_{n\to\infty}(nq - [nq]) \leq \limsup_{n\to\infty}(nq - [nq]) \leq 1$.
Use division ~~alog~~ algorithm, write ~~x_n~~ $q = \frac{a}{b}$ where a, b coprime.
$\exists k \in \mathbb{N}$ s.t. $ka \equiv b-1 \pmod{b}$. $\Rightarrow kq - [kq] = \frac{b-1}{b}$.
However, since $x - [x] < 1$ $\forall x$, kq must be a multiple of $\frac{1}{b}$,
(as well as $kq - [kq]$)
$\frac{b-1}{b}$ must be the supremum. $\limsup_{n\to\infty} x_n = \frac{b-1}{b}$.
For infimum, let $k = b$. it's obvious $\liminf_{n\to\infty} x_n = 0$.

3. a) let $\{z_n\} = \{x_n + y_n\}$. By B-W theorem, $\exists \{z_{n_k}\}$ that converges.
We construct $z_{n_k} = x_{n_k} + y_{n_k}$, and then extract $\{x_{n_{kj}}\}$ and $\{y_{n_{kj}}\}$ subsequences that converge.
We then extract $\{z_{n_{kjs}}\}$ from $\{z_{n_{kj}}\}$ (which we know exists).
Note $\lim_{n\to\infty} z_{n_{kjs}} = \lim_{n\to\infty} x_{n_{kjs}} + \lim_{n\to\infty} y_{n_{kjs}} = \lim_{n\to\infty} x_{n_{kj}} + \lim_{n\to\infty} y_{n_{kj}}$
$\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$ (by definition of limsup). The claim thus follows
also, limit of a convergent subsequence of a convergent sequence equals the original limit

b) let $\{x_n\} = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$, $\{y_n\} = \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$. We see $0 < 2$, the claim follows

c) $\limsup x_n = \sup E$, where E is the set of all subsequential limits of $\{x_n\}$.
let E_y correspond to y_n. Since $x_n \to x$, E_x is simply $\{x\}$.
$\because$ x_n converges to x, $\{x_{n_k} + y_{n_k}\}$ converges to p iff $\{y_{n_k}\}$ converges to $p - x$.
We know the subsequential limits' set is simply $x + E_y$.
Thus $\sup(x + E_y) = x + \sup(E_y) \Rightarrow \limsup(x_n + y_n) = \lim x_n\,(x) + \limsup y_n$.

d) We take $\lim_{n\to\infty}\sup(-x) = -\lim_{n\to\infty}\inf(x)$ for granted.

Then in a), $-\liminf_{n\to\infty}(-x_n-y_n) \leq -\liminf_{n\to\infty}(-x_n) - \liminf_{n\to\infty}(-y_n)$.

WLOG let $x_n = -x_n$, $y_n = -y_n$, we have $\liminf_{n\to\infty}(x_n+y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$.

4.

5.

6. a) $E = \{0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$. $\liminf_{n\to\infty} a_n = 0$, $\limsup_{n\to\infty} a_n = \frac{4}{7}$

b) $E = \{0, \frac{3}{2}\}$. $\liminf_{n\to\infty} a_n = 0$, $\limsup_{n\to\infty} a_n = \frac{3}{2}$.

c) $E = \{\pm e, \pm \frac{1}{\sqrt{2}} e, 0\}$ $\liminf_{n\to\infty} a_n = -\frac{e}{\sqrt{2}}$, $\limsup_{n\to\infty} a_n = \frac{e}{\sqrt{2}}$.

7. For any set $A \subset \mathbb{R}$ we know $\sup(-A) = -\inf(A)$

$\hookrightarrow$ thus, $\sup_{k\geq n}(-x_k) = -\inf_{k\geq n}(x_k)$ $\forall n \in \mathbb{N}$.

Thus, taking limits, $\limsup_{n\to\infty}(-x_n) = \lim_{n\to\infty}(\sup_{k\geq n}(-x_k)) = -\lim_{n\to\infty}(\inf_{k\geq n}(x_k)) = -\liminf_{n\to\infty}(x_n)$

8.

9. We construct $\{a_n\} = (1+\frac{1}{\lfloor x_n \rfloor})^{\lfloor x_n \rfloor}$, $\{c_n\} = (1+\frac{1}{\lceil x_n \rceil})^{\lceil x_n \rceil}$.

$a_n \leq (1+\frac{1}{\lfloor x_n \rfloor})^{\lfloor x_n \rfloor} \leq (1+\frac{1}{x_n})^{x_n} \leq (1+\frac{1}{\lceil x_n \rceil})^{\lceil x_n \rceil} \leq c$.

we know as $n\to\infty$, $\lfloor x_n \rfloor, \lceil x_n \rceil \to \infty$.

Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = e$.

Hence, $\lim_{n\to\infty}(1+\frac{1}{x_n})^{x_n} = e$.