

### Problem 1

Let  $a_0 := 2021$  and

$$a_{n+1} := \frac{1}{2} \left( a_n + \frac{1}{a_n} \right).$$

Show that  $a_n$  converges (in  $\mathbb{R}$ ) and find the limit.

We will show the following:

1.  $(a_n)$  is a non-increasing sequence;
2.  $(a_n)$  is bounded from below, therefore by Thm 5.1,  $\lim_{n \rightarrow \infty} a_n$  exists;
3.  $\lim_{n \rightarrow \infty} a_{n+1}$  also exists and it is equal to  $\lim_{n \rightarrow \infty} a_n$ , therefore by taking the limit of both sides and using the result from #8, we have the equation  $x = (x + x^{-1})/2$  that the limit  $x = \lim_{n \rightarrow \infty} a_n$  has to satisfy.

Observe that  $(a_n)$  is non-increasing if and only if  $a_{n+1} - a_n \leq 0$  for all  $n$ . Observe that

$$a_{n+1} - a_n = \frac{1 - a_n^2}{2a_n} \leq 0.$$

Clearly, for  $n = 0$ , the above inequality is true. To show that the inequality holds for all  $n \geq 0$ , we use induction. Observe that

$$a_n = \frac{a_{n-1}^2 + 1}{2a_{n-1}} \geq 1 \iff \frac{(a_{n-1} - 1)^2}{2a_{n-1}} \geq 0,$$

which is true provided that  $a_{n-1} > 0$ . So the entire sequence is greater than or equal to 1 provided that  $a_0 \geq 1$ , which is true. Hence we have shown that the sequence is non-increasing. Also, incidentally, we have shown that  $a_n \geq 1$  for every  $n$ , hence the sequence is also bounded from below. Thus by Thm 5.1,  $\lim_{n \rightarrow \infty} a_n = x$  exists and  $x \geq 1$  (by Thm. 5.8).

Let  $\epsilon > 0$  be given. Since  $(a_n)$  converges, it is Cauchy, hence there exists  $N$  such that for all  $n > N$ ,  $|a_{n+1} - a_n| < \epsilon/2$ . Also, since  $a_n \rightarrow x$ , there exists  $M$  such that for all  $n > M$ ,  $|a_n - x| < \epsilon/2$ . Thus for all  $n > \max\{N, M\}$ ,

$$|a_{n+1} - x| \leq |a_{n+1} - a_n| + |a_n - x| < \epsilon.$$

Therefore  $a_{n+1}$  also converges to  $x$ . Since  $a_n \geq 1$ , and  $x \geq 1$ , by #8,

$$\lim_{n \rightarrow \infty} a_{n+1} = x = \lim_{n \rightarrow \infty} \frac{1}{2} \left( a_n + \frac{1}{a_n} \right) = \frac{1}{2} \left( x + \frac{1}{x} \right) \iff x^2 - 1 = 0,$$

implying  $x = 1$  or  $x = -1$ . But since  $x \geq 1$ ,  $x = 1$ .

## Problem 2

Let  $q \in \mathbb{Q}$ . Find the set of limit points of the sequence  $(x_n) \subset \mathbb{R}$  where

$$x_n := nq - \lfloor nq \rfloor.$$

Find  $\limsup_{n \rightarrow \infty} x_n$  and  $\liminf_{n \rightarrow \infty} x_n$ .

There are two ways to do this problem, both of which I present here. Let us first assume that we have a rational  $p/q$  where  $p, q$  are coprime. That is,  $\gcd(p, q) = 1$ . Note we can assume without loss of generality that  $q > 0$ .

Both methods begin with a simple observation that for every integer  $p$  and a positive integer  $q$ , there exists a unique integer  $s$  and a non-negative integer  $r$  such that

$$p = sq + r,$$

and  $0 \leq r < q$ . Hence by dividing both sides by  $q$ , we obtain

$$\frac{p}{q} = s + \frac{r}{q}.$$

Here,  $s$  is equal to  $\lfloor p/q \rfloor$ , the largest integer not exceeding  $p/q$ . Hence

$$\frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor = \frac{r}{q},$$

where  $r$  is the remainder that results when  $np$  is divided by  $q$ . In other words, if  $n = kq + r$  for some  $k \in \mathbb{N}$ ,  $r \in \{0, 1, \dots, q-1\}$  then

$$x_n = \frac{rp}{q} - \left\lfloor \frac{rp}{q} \right\rfloor.$$

Note that the right-hand side does not depend on  $k$ . This way we have “separated”  $(x_n)$  into  $q$  convergent subsequences (as in Ex. 5.7.3 and in #6(a) below), and so the set of limit points is (recall (5.2))

$$E = \left\{ 0, \frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor, \frac{2p}{q} - \left\lfloor \frac{2p}{q} \right\rfloor, \dots, \frac{(q-1)p}{q} - \left\lfloor \frac{(q-1)p}{q} \right\rfloor \right\}.$$

Note that  $E$  (although not characterized explicitly) is a well-defined, finite subset of  $[0, 1)$ , given the rational number  $p/q$ . Hence (by Def. 5.4)

$$\liminf_{n \rightarrow \infty} x_n = \min E = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \max E.$$

(Note that we cannot deduce that  $\max E = \frac{(q-1)p}{q} - \left\lfloor \frac{(q-1)p}{q} \right\rfloor$ , as it is not clear that it is the largest element.)

### Problem 3

Let  $(x_n), (y_n) \subset \mathbb{R}$  be sequences bounded below.

(a) Show that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n.$$

(b) Given an example of a pair of sequences  $(x_n), (y_n)$  for which the inequality in (a) is sharp.

(c) Show that if  $(x_n)$  converges, then the inequality in (a) becomes an equality.

(d) State and prove an analogous claim about the lower limit.

(a) If one of the  $\limsup_{n \rightarrow \infty} x_n$  or  $\limsup_{n \rightarrow \infty} y_n$  is infinite, then the inequality holds trivially. Hence the only interesting case is when both  $X := \limsup_{n \rightarrow \infty} x_n$  and  $Y := \limsup_{n \rightarrow \infty} y_n$  are real numbers (note the case of  $-\infty$  is excluded by assumption).

Then the sequences  $(x_n)$  and  $(y_n)$  are bounded (by Thm. 5.5 (B), or by noting that otherwise limit superior would be  $\infty$ , which we assumed not). Let  $z_n := x_n + y_n$ . Let  $(z_{n_k})$  be a subsequence such that

$$\lim_{k \rightarrow \infty} z_{n_k} = \limsup_{n \rightarrow \infty} z_n =: Z.$$

Note such subsequence exists by Thm. 5.5(A). Since  $(x_n)$  is a bounded sequence, so is its subsequence  $(x_{n_k})$ . Hence by the Bolzano-Weierstrass Theorem (Thm. 4.9), there exists a subsequence  $(x_{n_{k_j}})_{j \geq 1}$  of  $(x_{n_k})_{k \geq 1}$  that converges. But whatever it converges to (say,  $\xi$ ), it is bounded above by  $X$ , by the definition of limit superior (it is the supremum of the set of limit points, recall Def. 5.4), so  $\lim_{j \rightarrow \infty} x_{n_{k_j}} \leq X$ . Similarly, the corresponding subsequence  $(y_{n_{k_j}})_{j \geq 1}$  is a bounded sequence, hence there exists a subsequence  $(y_{n_{k_{j_s}}})_{s \geq 1}$  that converges to something (say,  $\zeta$ ) that is bounded above  $Y$ . Then (because every subsequence of the convergent sequence  $(x_{n_{k_j}})_{j \geq 1}$  converges to the same limit  $\xi$ ) we can use Ex. 4.13.1 to obtain

$$\lim_{s \rightarrow \infty} (x_{n_{k_{j_s}}} + y_{n_{k_{j_s}}}) = \lim_{s \rightarrow \infty} x_{n_{k_{j_s}}} + \lim_{s \rightarrow \infty} y_{n_{k_{j_s}}} = \xi + \zeta \leq X + Y.$$

Similarly  $(z_{n_{k_{j_s}}})_{s \geq 1}$  converges to the same limit  $Z$  (as it is a subsequence of  $(z_k)_{k \geq 1}$ ), and so

$$Z \leq X + Y,$$

as required.

(b) To make the inequality sharp, observe that we can take the sequences  $(x_n) := (-1, 1, -1, 1, \dots)$  and  $(y_n) := (1, -1, 1, -1, \dots)$ . Then  $z_n = 0$  for all  $n$ , so

$\limsup(x_n + y_n) = 0$  (by Cor. 5.6.2), yet  $\limsup x_n = \limsup y_n = 1$ .

(c) If  $\lim_{n \rightarrow \infty} x_n = x$ , then for every subsequence  $(x_{n_k})$ ,  $x_{n_k} \rightarrow x$ . Let  $(y_{n_k})$  be a subsequence that converges to  $\limsup y_n$  (which we can specify thanks to Thm 5.5). Then

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \geq \lim_{k \rightarrow \infty} (x_{n_k} + y_{n_k}) = \lim_{k \rightarrow \infty} x_{n_k} + \lim_{k \rightarrow \infty} y_{n_k} = x + \limsup_{n \rightarrow \infty} y_n.$$

By (a) we already have the opposite inequality  $\limsup_{n \rightarrow \infty} (x_n + y_n) \leq x + \limsup_{n \rightarrow \infty} y_n$ , and so the claim follows.

(d) By #7 and part (a),

$$\begin{aligned} -\liminf_{n \rightarrow \infty} (x_n + y_n) &= \limsup_{n \rightarrow \infty} (-x_n + (-y_n)) \\ &\leq \limsup_{n \rightarrow \infty} (-x_n) + \limsup_{n \rightarrow \infty} (-y_n) \\ &= -\liminf_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} y_n. \end{aligned}$$

Multiplying by  $-1$  gives the inequality

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n.$$

#### Problem 4

Prove the analogue of Ex 4.13 and the squeeze theorem (Cor. 5.9) for infinite limits in  $\mathbb{R}$ , i.e., show that:

- (a) If  $x_n \rightarrow x \in \mathbb{R}$  and  $y_n \rightarrow +\infty$ , then  $x_n + y_n \rightarrow +\infty$ .
- (b) If  $x_n \rightarrow -\infty$  and  $y_n \rightarrow +\infty$ , then  $x_n y_n \rightarrow -\infty$ .
- (c) If  $x_n \rightarrow -\infty$  and  $m \in \mathbb{N}$ , then  $x_n^m \rightarrow +\infty$  (if  $m$  is even) or  $x_n^m \rightarrow -\infty$  (if  $m$  is odd).
- (d) If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$  and  $a_n \rightarrow \infty$ , then  $b_n \rightarrow \infty$ .

(a) Let  $M \in \mathbb{R}$  be given. Since  $(x_n)_{n \geq 1}$  there exists  $K > 0$  such that  $|x_n| \leq K$  for all  $n \geq 1$ . Since  $y_n \rightarrow +\infty$ , there exists  $N$  such that  $y_n > M + K$  for  $n \geq N$  (by Def. 4.21). Thus

$$x_n + y_n \geq y_n - |x_n| \geq M + K - K = M$$

for such  $n$ . That is  $x_n + y_n \rightarrow \infty$  (by Def. 4.21).

(b) Let  $M < 0$ . Since  $x_n \rightarrow -\infty$ , there exists  $N$  so that  $x_n \leq -1$  for  $n \geq N$ . Also there exists  $L$  so that  $y_n > -M$  for  $n \geq L$  (as  $y_n \rightarrow \infty$ ). Then for all  $n \geq \max\{N, L\}$ ,

$$x_n y_n \leq (-1) \cdot (-M) = M.$$

If  $M \geq 0$  then we proceed in the same way to note that  $x_n y_n \leq 0 \leq M$ . Thus  $x_n y_n \rightarrow -\infty$  by definition (Def. 4.21).

(c) First note that if  $x_n \rightarrow -\infty$  and  $y_n \rightarrow -\infty$  as  $n \rightarrow \infty$  then  $x_n y_n \rightarrow \infty$ , which can be proved in the same way as (b) (simply replace “ $y_n > -M$ ” by “ $y_n < M$ ”, and “ $x_n y_n \leq M$ ” by “ $x_n y_n \geq -M$ ”; and similarly, if  $M \geq 0$  replace  $x_n y_n \leq M$  by  $x_n y_n \geq 0 \geq -M$ ).

The claim now follows by induction (as in Ex. 4.13.4): Since  $x_n \rightarrow -\infty$ , the base case  $m = 1$  is true. If  $m$  is odd we have that  $m + 1$  is even and so  $x_n^{m+1} = x_n^m x_n \rightarrow \infty$  by the above fact. Of, otherwise,  $m$  is even then  $m + 1$  is odd and hence  $x_n^{m+1} = x_n^m x_n \rightarrow -\infty$  by (b).

(d) Since  $a_n \rightarrow \infty$ , for every  $M \in \mathbb{R}$ , there exists  $N$  such that for all  $n \geq N$ ,  $a_n > M$ . Since  $b_n \geq a_n$ , for the same  $N$ , we have  $b_n > M$  for every  $n \geq N$ .

### Problem 5

Let  $x_1 := 1$  and  $x_{n+1} := 2\left(2 - \frac{5}{x_n+3}\right)$  for  $n \geq 1$ .

- (a) Use induction to prove that  $x_n \in (0, 2)$  for all  $n \geq 1$ .
- (b) Deduce that  $x_n$  is non-decreasing.
- (c) Use Thm.5.1. to show that  $x_n$  converges, and find the limit.

The base case  $x_1 = 1 \in (0, 2)$  is evident. Assume that  $x_n \in (0, 2)$ . Then

$$\begin{aligned}0 < x_{n+1} < 2 &\iff -2 < 2\left(2 - \frac{5}{x_n+3}\right) - 2 < 0 \\ &\iff -1 < 1 - \frac{5}{x_n+3} < 0 \\ &\iff -2 < -\frac{5}{x_n+3} < -1 \\ &\iff 1 < \frac{5}{x_n+3} < 2 \\ &\iff \frac{5}{2} < x_n+3 < 5 \\ &\iff -\frac{1}{2} < x_n < 2.\end{aligned}$$

Hence  $x_{n+1} \in (0, 2)$  as desired (since the above equivalence gives that  $(x_n \in (0, 2))$  implies  $x_{n+1} \in (0, 2)$ )).

Note that  $x_n$  is non-decreasing iff  $x_{n+1} - x_n \geq 0$ . Observe:

$$x_{n+1} - x_n = 2\left(2 - \frac{5}{x_n+3}\right) - x_n = \frac{2 - x_n - x_n^2}{x_n+3} = -\frac{(x_n-2)(x_n+1)}{x_n+3}.$$

Since  $0 < x_n < 2$  by the previous part,  $x_{n+1} - x_n \geq 0$ .

By Thm 5.1,  $x := \lim_{n \rightarrow \infty} x_n$  exists and  $x \in [0, 2]$  (by Thm. 5.8). Also, as in #1,  $x = \lim_{n \rightarrow \infty} x_{n+1}$ . Hence by #8 and Ex. 4.13,

$$x = \lim_{n \rightarrow \infty} 2\left(2 - \frac{5}{x_n+3}\right) = 4 - \frac{10}{x+3}.$$

Solving for  $x$  gives  $x = -1$  or  $x = 2$ . Thus  $x = 2$  (as  $-1 \notin [0, 2]$ ).

### Problem 6

Find the set of limit points,  $\liminf_{n \rightarrow \infty}$  and  $\limsup_{n \rightarrow \infty}$  of the following sequences

$$(a) a_n := \frac{2n^2}{7} - \left[ \frac{2n^2}{7} \right],$$

$$(b) b_n := \frac{1}{2} \left( n - 2 - 3 \left[ \frac{n-1}{3} \right] \right) \left( n - 3 - 3 \left[ \frac{n-1}{3} \right] \right),$$

$$(c) c_n := \left( -1 - \frac{1}{n} \right)^n + \sin \frac{n\pi}{4}.$$

(a) Observe that if  $n = 7m + r$  for some  $0 \leq r \leq 6$ , then

$$2n^2 = 98m^2 + 28mr + 2r^2$$

and so

$$a_n = \frac{2n^2}{7} - \left[ \frac{2n^2}{7} \right] = 14m^2 + 4mr + \frac{2r^2}{7} - \left[ 14m^2 + 4mr + \frac{2r^2}{7} \right] = \frac{2r^2}{7} - \left[ \frac{2r^2}{7} \right],$$

since the integer part  $14m^2 + 4mr$  cancels. Thus we have “separated”  $a_n$  (as in #2 and in Ex. 5.7.3) into 7 constant sequences, depending on the remainder of  $n$  in the division by 7. Checking these 7 possibilities we obtain that each of the 7 subsequences equals (at each entry)  $0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}$ . Thus (recall (5.2))

$$E = \left\{ 0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7} \right\},$$

and so  $\liminf_{n \rightarrow \infty} a_n = 0$ ,  $\limsup_{n \rightarrow \infty} a_n = \frac{4}{7}$  (by Def. 5.4).

(b) Note that if  $n = 3k$  then  $[(n-1)/3] = k-1$ , if  $n = 3k+1$  or  $n = 3k+2$  then  $[(n-1)/3] = k$ . Thus

$$b_{3k} = \frac{1}{2} (3k - 2 - 3(k-1))(3k - 3 - 3(k-1)) = 0$$

$$b_{3k+1} = \frac{1}{2} (3k + 1 - 2 - 3k)(3k + 1 - 3 - 3k) = 1$$

$$b_{3k+2} = \frac{1}{2} (3k + 2 - 2 - 3k)(3k + 2 - 3 - 3k) = 0.$$

Thus  $E = \{0, 1\}$  and so  $\liminf_{n \rightarrow \infty} b_n = 0$ ,  $\limsup_{n \rightarrow \infty} b_n = 1$  (by Def. 5.4).

(c) Observe that

$$c_n = \left( -1 - \frac{1}{n} \right)^n + \sin \frac{n\pi}{4} = (-1)^n \left( 1 + \frac{1}{n} \right)^n + \sin \frac{n\pi}{4}.$$

Let  $n = 8k + r$  where  $m \in \{0, 1, \dots, 7\}$ . Note that then

$$c_{8k+r} = (-1)^r \left(1 + \frac{1}{8k+r}\right)^{8k+r} + \sin \frac{r\pi}{4},$$

which converges, as  $k \rightarrow \infty$ , to

$$(-1)^r e + \sin \frac{r\pi}{4}.$$

(Note here we used #9.) Thus, as above, we have separated  $c_n$  into 8 convergent subsequences, and so checking the 8 possible values of  $r$ , we see that the limit points are  $e, \frac{1}{\sqrt{2}} - e, 1 + e, -\frac{1}{\sqrt{2}} - e, -1 + e$ . Hence  $\limsup_{n \rightarrow \infty} c_n = 1 + e$  and  $\liminf_{n \rightarrow \infty} c_n = -\frac{1}{\sqrt{2}} - e$ .

### Problem 7

Show that

$$\limsup_{n \rightarrow \infty} (-x_n) = -\liminf_{n \rightarrow \infty} x_n.$$

Let  $E^+$  be the set of limit points of  $(x_n)$ , and let  $E^-$  be the set of limit points of  $(-x_n)$ . Then for each  $e \in E^+$ , there exists a subsequence  $(x_{n_k})$  that converges to  $e$ . Then  $(-x_{n_k}) \subseteq (-x_n)$  converges to  $-e$ . This establishes that  $-E^+ \subseteq E^-$ .

Let  $e'$  be a limit point of  $(-x_n)$  corresponding to the subsequence  $(-x_{n_k})$ . Then  $(x_{n_k})$  converges to  $-e'$ , so  $E^- \subseteq -E^+$ . This establishes that  $-E^+ = E^-$ .

In PS1, we showed that<sup>1</sup>  $\sup(-A) = -\inf A$ . Since  $\limsup(-x_n) = \sup(E^-) = \sup(-E^+) = -\inf E^+ = -\liminf x_n$ , we have the formula.

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<sup>1</sup>Technically, that exercise was for a *bounded* set  $A$ , but the formula holds trivially when  $\sup A = \infty$ .

### Problem 8

Let  $(x_n)$  be a sequence of positive real numbers. Show that

$$\liminf_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{\limsup_{n \rightarrow \infty} x_n}.$$

Deduce that if  $x_n \rightarrow x$  for some  $x \in \mathbf{R} \setminus \{0\}$ , then  $1/x_n \rightarrow 1/x$ .

Assume first that  $\limsup_{n \rightarrow \infty} x_n = \infty$ . Then there exists a subsequence  $(x_{n_k})$  such that  $x_{n_k} \rightarrow \infty$  as  $k \rightarrow \infty$ . Since

$$0 \leq \frac{1}{x_{n_k}} \leq \frac{1}{[x_{n_k}]},$$

and since  $(1/[x_{n_k}])_{k=1}^{\infty}$  is a subsequence of  $(1/n)$ , which converges to 0, by the squeeze theorem,  $1/x_{n_k} \rightarrow 0$ . Thus we have  $\liminf_{n \rightarrow \infty} 1/x_n = 0$  (because the sequences are bounded below by 0).

Assume now  $\limsup_{n \rightarrow \infty} x_n = 0$ . Since  $x_n$  is a positive sequence, we must have (by Thm. 5.8)

$$0 \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n = 0 \iff \liminf_{n \rightarrow \infty} x_n = 0,$$

by Cor. 5.6.2. Hence the sequence converges to 0 if  $\limsup_{n \rightarrow \infty} x_n = 0$ . Then for every  $m \in \mathbf{N}$ , there exists  $N_m$  such that for every  $n \geq N_m$ ,

$$0 < x_n < \frac{1}{m} \implies \frac{1}{x_n} > m.$$

Thus  $1/x_n \rightarrow \infty$  by Def. 4.21, and so in particular  $\liminf_{n \rightarrow \infty} 1/x_n = \infty$ .

Finally, assume  $\limsup_{n \rightarrow \infty} x_n = a > 0$ . Then there exists a subsequence  $(x_{n_k})$  converging to  $a$ , and so (by Def. 3.4) for every  $m \in \mathbf{N}$  with  $0 < m^{-1} < a/2$ , there exists  $N_m$  such that for all  $n \geq N_m$ ,  $|x_n - a| < m^{-1}$ , or equivalently, for all  $n \geq N_m$ ,

$$a - \frac{1}{m} < x_n < a + \frac{1}{m} \implies \frac{1}{a + 1/m} < \frac{1}{x_n} < \frac{1}{a - 1/m}.$$

Hence we may choose an increasing sequence of natural numbers  $(N_m)_{m=1}^{\infty}$  such that

$$\frac{1}{a + 1/m} < \frac{1}{x_{N_m}} < \frac{1}{a - 1/m}.$$

Observe that

$$\left| \frac{1}{a + 1/m} - \frac{1}{a} \right| = \left| \frac{-1/m}{a(a + 1/m)} \right| < \left| \frac{1/m}{2a} \right| \rightarrow 0$$

as  $m \rightarrow \infty$ , and similarly,

$$\left| \frac{1}{a - 1/m} - \frac{1}{a} \right| = \left| \frac{1/m}{a(a - 1/m)} \right| < \left| \frac{1/m}{a/2} \right| \rightarrow 0.$$

Hence by the squeeze theorem (Cor. 5.9),  $x_{N_m}^{-1} \rightarrow 1/a$  as  $m \rightarrow \infty$ . This establishes that  $1/a$  is a limit point of the sequence  $(1/x_n)$ , i.e. part (A) of Thm. 5.5 is satisfied. As for part (B) take any  $r \in (0, 1/a)$  then  $1/r > a$  and so there exists  $N \in \mathbb{N}$  such that  $x_n \leq 1/r$  for  $n \geq N$ . Thus also  $1/x_n \geq r$  and so part (B) is also satisfied. Thus the claim follows from Thm. 5.5.

If  $x_n \rightarrow x \neq 0$ , then

$$\liminf_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{\limsup_{n \rightarrow \infty} x_n} = \frac{1}{\lim_{n \rightarrow \infty} x_n}.$$

Similarly,

$$\limsup_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{\liminf_{n \rightarrow \infty} x_n} = \frac{1}{\lim_{n \rightarrow \infty} x_n} = \liminf_{n \rightarrow \infty} \frac{1}{x_n} = \lim_{n \rightarrow \infty} \frac{1}{x_n}.$$

This gives the desired equality  $\lim_{n \rightarrow \infty} 1/x_n = 1/\lim_{n \rightarrow \infty} x_n$ .

### Problem 9

Let  $(x_n) \subset (0, \infty)$  be a sequence such that  $x_n \rightarrow \infty$ . Prove that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{x_n}\right)^{x_n} = e.$$

Evidently if  $(x_n)$  is a sequence of positive integers, then it is a subsequence of  $(n)_{n=1}^{\infty}$ , and since  $(1 + 1/n)^n \rightarrow e$ , we also have  $(1 + 1/x_n)^{x_n} \rightarrow e$ . In a similar fashion, for any integer sequence  $(x_n)$  diverging to infinity,  $(1 + 1/x_n)^{1+x_n} \rightarrow e$  as well.

Observe that since  $\lfloor x_n \rfloor + 1 > x_n \geq \lfloor x_n \rfloor$ ,

$$\left(1 + \frac{1}{\lfloor x_n \rfloor + 1}\right)^{\lfloor x_n \rfloor} \leq \left(1 + \frac{1}{x_n}\right)^{x_n} \leq \left(1 + \frac{1}{\lfloor x_n \rfloor}\right)^{\lfloor x_n \rfloor + 1},$$

and since  $(\lfloor x_n \rfloor)$  is an integer sequence converging to infinity, by the above observation, the rightmost side converges to  $e$ . Also,

$$\left(1 + \frac{1}{\lfloor x_n \rfloor + 1}\right)^{\lfloor x_n \rfloor} = \left(1 + \frac{1}{\lfloor x_n \rfloor + 1}\right)^{-1} \left(1 + \frac{1}{\lfloor x_n \rfloor + 1}\right)^{\lfloor x_n \rfloor + 1} \rightarrow e$$

(as  $(1 + 1/(\lfloor x_n \rfloor + 1))_{n=1}^{\infty}$  is a subsequence of  $(1 + 1/n)$ , and we used #8 to deduce that the first factor on the right-hand side converges to 1).

### Problem 10

Prove that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{m}{n}\right)^{-n} = e^m.$$

Let  $x_n := n/m$  for  $m \in \mathbf{N}$ . Then  $x_n \rightarrow \infty$ , and

$$\left(1 - \frac{m}{n}\right)^{-n} = \left(\left(1 - \frac{1}{x_n}\right)^{-x_n}\right)^m.$$

By #9 and 4.13.4, this sequence then converges to  $e^m$ .

### Problem 11

Show that  $\sqrt[n]{n} \rightarrow 1$  as  $n \rightarrow \infty$ .

Let  $x_n := \sqrt[n]{n} - 1$ . Since  $n \geq 1$ , we have

$$n = (1 + x_n)^n \geq 1 + \frac{n(n-1)}{2} x_n^2 \implies x_n \leq \frac{\sqrt{2}}{\sqrt{n}}$$

by the binomial theorem. Evidently  $\sqrt[n]{n} \geq 1$ , so we have

$$0 \leq x_n \leq \frac{\sqrt{2}}{\sqrt{n}},$$

and since  $\sqrt{2}/\sqrt{n} \rightarrow 0$  as  $n \rightarrow \infty$  (given  $\epsilon > 0$ , take  $N > 2/\epsilon^2$ ), by the squeeze theorem (Cor. 5.9),  $x_n \rightarrow 0$ .