Math 425A Problem Sheet 6 (due 9am on Wednesday, 13th Oct)

Essential problems

1. (4 pts) Suppose that (x_n) is a sequence such that

$$\lim_{n\to\infty}((x_1+1)(x_2+1)\dots(x_n+1))=y,$$

where $y>0$ or $y=+\infty$. Show that

$$\sum_{n\geq 1}\frac{x_n}{(x_1+1)(x_2+1)\dots(x_n+1)}=1-\frac{1}{y},$$

where we use the convention that $\frac{1}{+\infty}=0$. *(Hint: Use the fact that $x_n=(x_n+1)-1$ to obtain an explicit formula for S_n (the partial sum).)* Use this fact to find the sum of

$$\sum_{n\geq 1}\frac{2n-1}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}.$$

2. (3 pts) Show that

$$\sum_{n\geq 1}\frac{1}{(2n-1)^2}=\frac{3}{4}\sum_{n\geq 1}\frac{1}{n^2}.$$

3. (3 pts) Determine whether the series

$$\sum_{n\geq 1}\frac{n^n}{\mathrm{e}^n n!}$$

converges. *(Hint: use question 5 below and the fact that (y_n) from Ex. 5.3 is decreasing.)*

Additional problems

4. (1 pt) For each of the following series determine whether or not it converges.

(a) $\sum_{n\geq 1}\left(\frac{n}{n+1}\right)^{n(n+1)}$,
(b) $\sum_{n\geq 1}\left(1-\cos\frac{1}{n}\right)$,
(c) $\sum_{n\geq 1}\frac{1}{n^2-\log n}$.

(Hint: Recall that $1-\cos 2x=2\sin^2 x$, and that $\sin x,\log x\leq x$ for $x>0$.)

5. (1 pt) Let $\sum a_n$ and $\sum b_n$ be series of positive terms such that for some $N\in\mathbb{N}$

$$\frac{a_{n+1}}{a_n}\leq\frac{b_{n+1}}{b_n}\qquad\text{for } n\geq N.$$

Show that convergence of $\sum b_n$ implies convergence of $\sum a_n$. *(Hint: Note that a_n/b_n is a nonincreasing sequence, and use the comparison test, Lem. 6.4.2.)* Explain why we cannot use the ratio test in this question (i.e. the fact that $\limsup_{n\to\infty}a_{n+1}/a_n\leq\limsup_{n\to\infty}b_{n+1}/b_n$).

6. (1 pt) For which values of $\alpha\in\mathbb{R}$ does the series

$$\sum_{n\geq 2}\frac{1}{n(\log n)^\alpha}$$

converge? *(Hint: Use Cauchy condensation test and the fact that $\log x$ is an increasing function.)*

7. (1 pt) Let $\sum_{n\geq 1} a_n$ be a convergent series of positive elements. Show that then

$$\sum_{n\geq 1} \frac{a_1+\ldots+a_n}{n}$$

diverges.

8. (1 pt) Show that the series

$$\sum_{n\geq 1} (-1)^n \left(\mathrm{e} - \left(1 + \frac{1}{n} \right)^n \right)$$

converges.

9. (1 pt) For which values of $\alpha \in \mathbb{R}$ does the series

$$\sum_{n\geq 1} \left(\frac{\alpha n}{n+1} \right)^n$$

converge? For which $\alpha \in \mathbb{R}$ does it converge absolutely?

10. (1 pt) Show that the series

$$\sum_{n\geq 1} (-1)^n \sin \frac{\alpha}{n}$$

converges for any $\alpha \in \mathbb{R}$, but it converges absolutely only for $\alpha = 0$.

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