MATH 425a PS 6

1\.

let $S_n = \sum_{k=1}^{n} \frac{x_k+1-1}{(x_1+1)\cdots(x_k+1)} = 1 - \frac{1}{x_1+1} + \frac{1}{x_1+1} - \frac{1}{(x_1+1)(x_2+1)} + \cdots + \frac{1}{\prod_{i=1}^{n-1}(x_i+1)} - \frac{1}{\prod_{i=1}^{n}(x_i+1)}$

$\hookrightarrow = 1 - \frac{1}{(x_1+1)\cdots(x_n+1)}$

Thus since $\lim_{n\to\infty}((x_1+1)\cdots(x_n+1)) = y$, $\lim_{n\to\infty} = 1 - \frac{1}{y}$.

$\sum_{n=1}^{\infty} \frac{2n-1}{2\cdot 4\cdot 6\cdots 2n} \Rightarrow$ let $x_n = 2n-1$. ~~This~~ It equals $1 - \frac{1}{\infty} = 1 - 0 = 1$.

2\.

We know $\sum_{n=1} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$

Also, $\frac{1}{4}\sum_{n=1} \frac{1}{n^2} = \frac{1}{2^2} + \frac{1}{2^2 2^2} + \cdots + \frac{1}{2^2 n^2} = \frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2}$ (n arbitrary).

Thus $\frac{3}{4}\sum_{n=1} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} = \sum_{n=1} \frac{1}{(2n-1)^2}$, as desired.

3\.

let $b_n = \frac{n^{n+1}}{e^n n!}$. $\frac{b_{n+1}}{b_n} = \frac{(n+1)^{n+2} e^n n!}{n^{n+1} e^{n+1} (n+1)!} = (1+\frac{1}{n})^{n+1} \frac{1}{e} \frac{n+1}{n+1} = (1+\frac{1}{n})^{n+1} \frac{1}{e}$.

we take for granted that $(1+\frac{1}{n})^{n+1}$ decreasing sequence that $\to e$.

Thus $(1+\frac{1}{n})^n \frac{1}{e} > e \cdot \frac{1}{e} = 1$. $\therefore$ ~~b_n increasing~~ b_n increasing.

$\hookrightarrow \frac{n^n}{e^n n!} \geq \frac{1}{e^n}$ as $\sum_{n=1} \frac{n^{n+1}}{e^n n!} > \sum_{n=1} \frac{n}{e^n n!}$.

we know $\sum \frac{1}{e^n}$ diverges, so $\sum \frac{n^n}{e^n n!}$ diverges as well.

6\.

$0 < \frac{1}{(n+1)(\log(n+1))^a} < \frac{1}{n(\log n)^a}$. By Cauchy condensation test,

$\sum 2^k a_{2^k} = \sum 2^k \cdot \frac{1}{2^k \log(2^k)^a} = \frac{1}{(\log 2)^a} \sum \frac{1}{k^a}$. we know $a \in \mathbb{R}$, $\therefore a > 1$.

7\.

$\sum_{n=1} \frac{a_1 + \cdots + a_n}{n} > \sum_{n=1} \frac{1}{n} a_1 \Rightarrow$ this diverges as $\sum_{n=1} \frac{1}{n}$ diverges. Thus $\sum_{n=1} \frac{a_1+\cdots+a_n}{n}$ diverges.

8\.

Invoking the alternating series test. let $b_n = e - (1+\frac{1}{n})^n$. $0 < (1+\frac{1}{n})^n < e$.

We know $(1+\frac{1}{n})^n$ monotonically increasing and $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$,

so $\lim_{n\to\infty} b_n = e - e = 0$. Thus $\sum_{n=1} (-1)^n (e - (1+\frac{1}{n})^n)$ converges.