Solutions to Problem Set 6

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1. Observe that

$$\begin{aligned}\frac{x_n}{(x_1+1)(x_2+1)\cdots(x_n+1)} &= \frac{(x_n+1)-1}{(x_1+1)(x_2+1)\cdots(x_n+1)} \\ &= \frac{x_n+1}{(x_1+1)(x_2+1)\cdots(x_n+1)} - \frac{1}{(x_1+1)(x_2+1)\cdots(x_n+1)} \\ &= \frac{1}{(x_1+1)(x_2+1)\cdots(x_{n-1}+1)} - \frac{1}{(x_1+1)(x_2+1)\cdots(x_n+1)}\end{aligned}$$

Let us call $a_n = \frac{1}{(x_1+1)(x_2+1)\cdots(x_n+1)}$ with the convention that $a_0 = 1$. Then the terms of our series look like $(a_{n-1} - a_n)$, Therefore the partial sums S_m of the series can be computed very easily, as

$$S_m = \sum_{n=1}^{n=m}(a_{n-1} - a_n) = a_0 - a_m = 1 - a_m$$

Recall that the sum of a series is the limit of the sequence of partial sums. Therefore,

$$\sum_{n\geq 1} \frac{x_n}{(x_1+1)(x_2+1)\cdots(x_n+1)} = \lim_{m\to\infty} S_m = \lim_{m\to\infty}(1-a_m) = 1 - \frac{1}{y}$$

as by the given condition on the x_n's the limit of a_n is $\frac{1}{y}$.

For the special case, we set $x_n = 2n-1$ in the above calculation and note that

$$(x_1+1)(x_2+1)\cdots(x_n+1) = 2\cdot 4\cdot 6\ldots 2n \xrightarrow[n\to\infty]{} +\infty$$

So, applying the above result we get that the limit is $1 - \frac{1}{\infty} = 1$.

2. Let $S = \sum\limits_{n\geq 1} \frac{1}{n^2}$. We now separate the odd and even terms,

$$\begin{aligned} S &= \sum_{n\geq 1} \frac{1}{n^2} \\ &= \sum_{k\geq 1} \frac{1}{(2k)^2} + \sum_{k\geq 1} \frac{1}{(2k-1)^2} \\ &= \frac{1}{4}\sum_{k\geq 1} \frac{1}{k^2} + \sum_{k\geq 1} \frac{1}{(2k-1)^2} \\ &= \frac{1}{4}S + \sum_{k\geq 1} \frac{1}{(2k-1)^2} \end{aligned}$$

which gives us that $\sum\limits_{k\geq 1} \frac{1}{(2k-1)^2} = \frac{3}{4}S$, as required.

Note here that we are allowed to subtract $\frac{1}{4}S$ from both sides of the equation as S is a finite quantity (i.e, the series $\sum\limits_{n\geq 1} \frac{1}{n^2}$ converges). Also, we are allowed to split the sequence into two parts as in the second line because absolutely convergent series can be rearranged.

3. We'll follow the hint for this question. From Ex. 5.3 we know what $(1 + \frac{1}{n})^{n+1} \geq \mathrm{e}$ for all $n \geq 1$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n}\right)^n}{\mathrm{e}} \geq \frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{1}{n}\right)^{n+1}} = \frac{n}{n+1} = \frac{\frac{1}{n+1}}{\frac{1}{n}} =: \frac{b_{n+1}}{b_n}$$

for every $n \geq 1$. Suppose that $\sum a_n$ converges. Then problem 5 implies that $\sum b_n$ converges as well, which contradicts Ex. 6.8 (as $\sum 1/n$ is a divergent series). Thus $\sum a_n$ diverges.

4. (a) We apply the root test (Thm. 6.10.1). Note that $\left(\frac{n+1}{n}\right)^n \to e$ as n approaches infinity. This implies $\left(\frac{n+1}{n}\right)^{n+1} \to e$ too because $\frac{n+1}{n} \to 1$. This implies $\left(\frac{n}{n+1}\right)^{n+1} \to \frac{1}{e}$. Since $\frac{1}{e} < 1$ we conclude that the series converges.

   (b) Note that $1 - \cos\frac{1}{n} = 2\sin^2\frac{1}{2n} \leq \frac{1}{2n^2}$ (as $\sin x \leq x$ for $x > 0$). Thus the series is convergent by the comparison test (Lem. 6.4.2) and Ex. 6.8 (i.e. $\sum 1/n^2$ is a convergent series).

   (c) Since $\log n \leq n$, We have that $\frac{1}{n^2 - \log n} \leq \frac{1}{n^2 - n} = \frac{1}{n(n-1)} \leq \frac{1}{(n-1)^2}$. Thus the series converges by the comparison test (Lem. 6.4.2) as $\sum 1/(n-1)^2$ converges (by Ex. 6.8).

5. We follow the hint. By the given condition, we have that $\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n}$ for all $n > N$ implying that the sequence $\left\{\frac{a_n}{b_n}\right\}_{n>N}$ is a nonincreasing sequence, and so in particular

$$\frac{a_n}{b_n} \leq \frac{a_N}{b_N} \qquad \text{for all } n \geq N$$

Hence

$$\sum_{n\geq N} a_n \leq \frac{a_N}{b_N} \sum_{n\geq N} b_n < \infty,$$

and so $\sum_{n>1} a_n < \infty$ as we are only adding finitely many terms.

The ratio test cannot be used here as it is not clear whether $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. We could use the ratio test if we knew, for example, that $\limsup_{n\to\infty} \frac{b_{n+1}}{b_n} < 1$.

6. First we show that no non-positive value of α works. Indeed, if $\alpha \leq 0$ then

$$\sum_{n\geq 1} \frac{1}{n(\log n)^\alpha} = \sum_{n\geq 1} \frac{(\log n)^{|\alpha|}}{n}. \tag{1}$$

Now for $n > 10$ we have $\log n > 1$, which implies

$$\sum_{n\geq 11} \frac{(\log n)^{|\alpha|}}{n} > \sum_{n\geq 11} \frac{1}{n}$$

Therefore using the comparison test (Lem. 6.4.3) and equation (1) we can conclude that the series diverges. Hence, we can assume that α is positive.

If α is positive then $n(\log n)^\alpha$ is an increasing function of n, making $\frac{1}{n(\log n)^\alpha}$ a decreasing function which allows us to apply the Cauchy condensation test. Let us look at the condensed sequence

$$\sum_{k\geq 1} 2^k \frac{1}{2^k(\log 2^k)^\alpha} = \sum_{k\geq 1} \frac{1}{k^\alpha(\log 2)^\alpha} = \frac{1}{(\log 2)^\alpha} \sum_{k\geq 1} \frac{1}{k^\alpha}$$

which converges only if $\alpha > 1$. Since the Cauchy condensation test is an if and only if condition we conclude that our original series too converges only if $\alpha > 1$.

7. Note that

$$\sum_{n\geq 1} \frac{a_1 + \ldots + a_n}{n} > a_1 \sum_{n\geq 1} \frac{1}{n}.$$

Thus the given series diverges (by the comparison test, Lem. 6.4.3) as $\sum 1/n$ does (Ex. 6.8).

8. By Ex. 5.3 the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n\geq 1}$ is an increasing sequence and with the limit e. Therefore, the sequence defined by $b_n := \left(e - \left(1 + \frac{1}{n}\right)^n\right)$ is a decreasing sequence and has limit equal to 0. Thus the alternating series test (Cor. 6.19) $\sum_{n\geq 1}(-1)^n b_n$ is convergent.

9. We first apply the Root test, calling $a_n = \left(\frac{\alpha n}{n+1}\right)^n$, we observe that

$$\limsup \sqrt[n]{|a_n|} = \limsup \frac{|\alpha| n}{n+1} = |\alpha| \tag{2}$$

So if $|\alpha| < 1$, we have that our series converges absolutely and hence conditionally too. Also if $|\alpha| > 1$, then our series is divergent (hence not conditionally convergent too). So the only two cases left to check are when $|\alpha| = 1$ i.e, when $\alpha = \pm 1$.

However in those two cases we have that $\lim_{n\to\infty} a_n \neq 0$. This is because if $\alpha = 1$ then $\lim_{n\to\infty} a_n = \frac{1}{e}$ and when $\alpha = -1$ the limit does not exist.

10. If $\alpha \geq 0$ then there exists $N \in \mathbb{N}$ such that $\frac{\alpha}{n} < 1$ for all $n > N$. Recall that in the interval $[0, 1]$ $\sin x$ is an increasing function. So $\left\{\sin \frac{\alpha}{n}\right\}_{n>N}$ is a decreasing sequence. The limit of this sequence is 0, so the alternating series test (Cor. 6.19) gives that $\sum\limits_{n>N} (-1)^n \sin \frac{\alpha}{n}$ is convergent. Therefore so is $\sum\limits_{n>1} (-1)^n \sin \frac{\alpha}{n}$ (as we are just appending finitely many terms in the front).

If $\alpha < 0$ then

$$\sum_{n\geq 1} (-1)^n \sin \frac{\alpha}{n} = \sum_{n\geq 1} (-1)^{n+1} \sin \frac{|\alpha|}{n} = -\sum_{n\geq 1} (-1)^n \sin \frac{|\alpha|}{n},$$

which converges by the same argument (as $|\alpha| > 0$).

For absolute convergence, note that $\left|(-1)^n \sin \frac{\alpha}{n}\right| = \sin \frac{|\alpha|}{n}$. So we have to show that $\sum\limits_{n\geq 1} \sin \frac{|\alpha|}{n}$ is divergent for all $\alpha \neq 0$. Recall that if $\alpha \neq 0$ then $\frac{\sin \frac{|\alpha|}{n}}{\frac{|\alpha|}{n}} \xrightarrow[n\to\infty]{} 1$. Therefore there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $\sin \frac{|\alpha|}{n} > \frac{1}{2} \frac{|\alpha|}{n}$, and so

$$\sum_{n>N} \sin \frac{|\alpha|}{n} > \frac{|\alpha|}{2} \sum_{n>N} \frac{1}{n}$$

Thus by the comparison test (Lem. 6.4.3) $\sum\limits_{n>N} \sin \frac{|\alpha|}{n}$ is divergent and hence so is $\sum\limits_{n>1} \sin \frac{|\alpha|}{n}$.

If $\alpha = 0$, then all terms are zero so absolute convergence holds trivially.