Math 425A Problem Sheet 7 (due 9am on Monday, 18th Oct)

Essential problems

1. (4 pts) Let $f\colon \mathbb{R} \to \mathbb{R}$.

(a) Show that

$$f \text{ is continuous } \Leftrightarrow f^{-1}((-\infty, c)) \text{ and } f^{-1}((c, \infty)) \text{ are open for every } c \in \mathbb{R}.$$

(Comment: In other words, in the particular case of the mapping from $\mathbb{R}$ into itself, f is continuous iff f is an open mapping on open half-lines; also hint: try mimicking the proof of Thm. 7.8)

(b) Deduce that the same equivalence holds with the intervals "$(-\infty, c)$", "(c, ∞)" replaced by "$(-\infty, c]$", "$[c, \infty)$", respectively, and "open" replaced by "closed". *(Hint: Use Cor. 7.9)*

(c) Deduce that

$$f \text{ is continuous } \Rightarrow f^{-1}(\{c\}) \text{ is closed for every } c \in \mathbb{R}.$$

(Hint: Note that $\{c\} = (-\infty, c] \cap [c, +\infty)$ and use Thm. 3.12.1 and question 5 below.) Also, show that the opposite implication is false (find a counterexample).

2. (3 pts) For each of the following conditions give an example of $f\colon \mathbb{R} \to \mathbb{R}$ (simply sketch the plot of f, no need for a proof) that satisfies it

(a) $\forall_x \forall_{\varepsilon>0} \forall_{\delta>0} \forall_y \;\; |x-y| \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon,$

(b) $\forall_x \forall_{\varepsilon>0} \exists_{\delta>0}$ s.t. $\forall_y \;\; |x-y| \le \varepsilon \Rightarrow |f(x) - f(y)| \le \delta,$

(c) $\forall_x \forall_{\varepsilon>0} \exists_{\delta>0}$ s.t. $\forall_y \;\; |f(x) - f(y)| > \varepsilon \Rightarrow |x-y| > \delta,$

(d) $\forall_x \exists_{\varepsilon>0}$ s.t. $\forall_{\delta>0} \forall_y \;\; |x-y| \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon,$ but f is discontinuous,

(e) f is continuous, but it is not true that $\forall_x \forall_{\varepsilon>0} \exists_{\delta>0}$ s.t. $\forall_y \;\; x-y \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon,$

(f) same as (e), but with "f is continuous" replaced by $\forall_x \forall_{\varepsilon>0} \exists_{\delta>0}$ s.t. $\forall_y \;\; x-y \le \delta \Rightarrow f(x) - f(y) \le \varepsilon.$

3. (3 pts) Let $f\colon X \to Y$ be continuous and let $E \subset X$ be dense in X.

(a) Show that $f(E)$ is dense in $f(X)$.

(b) Deduce that if $f, g\colon X \to Y$ are two continuous functions such that $f(x) = g(x)$ for every $x \in E$, then $f = g$. *(Comment: in other words, in order to uniquely determine a continuous function, it suffices to specify it on a dense subset.)*

(c) Use (b) to deduce that the function

$$f(x) := \begin{cases} x & x \text{ is irrational,} \\ 0 & x \text{ is rational} \end{cases}$$

and the function in question 7(a) below are not continuous.

Additional problems

4. (1 pt) Find the limit, or prove that it does not exist.

(a) $\lim_{x\to 0} \frac{1}{x} \cos x,$

(b) $\lim_{x\to 0} \frac{\sqrt{1+3x+x^2}-1}{x},$

(c) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$. *(Comment: You can use the fact that* $\lim_{x\to 0}\frac{\sin x}{x}=1$.*)*

5. (1 pt) Let $f\colon X\to Y$ be any function. Show the following properties of the preimage:

(a) $f^{-1}\left(\bigcup_\alpha V_\alpha\right)=\bigcup_\alpha f^{-1}(V_\alpha)$ for any union $\cup_\alpha V_\alpha\subset Y$ of subsets of Y.

(b) $f^{-1}(U^c)=(f^{-1}(U))^c$ for any $U\subset Y$ *(Comment: recall we have used it in the proof of Cor. 7.9).*

(c) Deduce from (b) that (a) also holds with the union "$\bigcup_\alpha$" replaced by the intersection "$\bigcap_\alpha$".

6. (1 pt) Show that the definition of continuity (Def. 7.5) is equivalent to the one where the "$\leq$" inequalities are replaced by the "$<$" inequalities. Namely, show that, given $f\colon X\to Y$ is such that

$$\forall_{\varepsilon>0}\exists_{\delta>0}\quad d_Y(f(x),f(y))\leq\varepsilon\quad\text{ if } d_X(x,y)\leq\delta$$

then also

$$\forall_{\varepsilon>0}\exists_{\delta>0}\quad d_Y(f(x),f(y))<\varepsilon\quad\text{ if } d_X(x,y)<\delta,$$

and vice versa.

7. (1 pt) Find the points $x\in\mathbb{R}$ at which f is continuous.

(a) $f\colon\mathbb{R}\to\mathbb{R}$, $f(x):=\begin{cases}x^2-1 & x\text{ is irrational,}\\ 0 & x\text{ is rational,}\end{cases}$

(b) $f\colon[0,1]\to\mathbb{R}$, $f(x):=\sup\{1-x^n\colon n\in\mathbb{N}\}$.

8. (1 pt) Let $f\colon\mathbb{R}\to\mathbb{R}$, and fix $x\in\mathbb{R}$. Show that if f is continuous at x then

$$\lim_{h\to 0}\left(f(x+h)-f(x-h)\right)=0$$

(hint: add and substract $f(x)$), but that the opposite implication is false (i.e. find a counterexample).

9. (1 pt) Let $f,g\colon\mathbb{R}\to\mathbb{R}$ be continuous functions. Show that $|f|$, $\max(f,g)$, $\min(f,g)$ are continuous. *(Hint: Use PS1.4 and PS2.7)*

10. (1 pt) Let $f\colon X\to Y$ be continuous let $Z(f):=\{x\in X\colon f(x)=0\}$ denote the zero set (or the null space or kernel) of f. Show that $Z(f)$ is closed.

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