MATH 425a PS 7

1.

a) We first show the forward direction.

Assume f is continuous. Then $\forall \varepsilon>0, \exists \delta>0$ s.t. $d_Y(f(x), f(p))<\varepsilon$ when $d_X(x,p)<\delta$, $p\in \overset{R}{\cancel{X}}$.

We know $V := (-\infty, c)\cup(c,\infty)$ open $\Rightarrow \exists \varepsilon>0$ s.t. $N_\varepsilon(f(x))\subset V$.

This means if $d_X(x,p)<\delta$, $\Rightarrow f(p)\in N_\varepsilon(f(x)) \Rightarrow f(p)\in V$ and $p\in f^{-1}(V)$.

Thus $N_\delta(x)\subset f^{-1}(V)$, meaning $f^{-1}((-\infty,c))$ and $f^{-1}((c,\infty))$ open.

We then show the backward direction.

Let V be the same subset of R. Assume $f^{-1}(V)$ is open, fix $\varepsilon>0$ $\exists\delta>0$ s.t.

$\cdot$ $d_Y(f(\bar{x}), f(x))<\varepsilon$ whenever $d_X(x,\bar{x})<\delta \Rightarrow$ ~~meaning $\bar{x}$~~ also, $\bar{x}\in N_\varepsilon(x)\subset f^{-1}(V)$.

Thus f is continuous.

b) $(-\infty, c]$ and $[c,\infty)$ are closed sets. Invoking theorem 7.9, having assumed f is continuous, $f^{-1}((-\infty,c])$ and $f^{-1}([c,\infty))$ must be closed too — and vice versa (i.e. $f: X\to Y$ continuous iff preimage of closed sets are closed).

c) Since f is continuous, $f^{-1}([c,\infty))$, $f^{-1}((-\infty,c])$ closed.

Then $f^{-1}([c,\infty))$ ~~an~~ $\cap f^{-1}((-\infty,c])$ closed. $\Rightarrow f([c,\infty)\cap(-\infty,c])$ closed.

$f(\{c\})$ is closed, as thus follows.

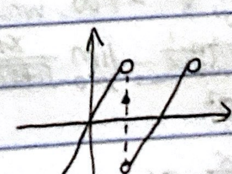
2.

a) $f(x)=5$, ~~$x\in R$~~

b) $f(x)=x$

c) $f(x)=x$

d) Something like this $\rightarrow$



e) ~~$f(x)=x^2$, $x\in$~~ $f(x)=x$

f) $f(x)=x$

3.

a) since f is continuous, ~~$\exists x_0 \epsilon$~~ $\forall x\ \varepsilon>0$, $\exists \delta>0$ s.t. $d_X(x,p)<\delta \Rightarrow d_Y(f(x),f(p))<\varepsilon$.

$\because E$ dense in ~~S~~ X, $\therefore \exists \bar{x}\in E$ s.t. $d_X(x,\bar{x})<\delta$ (any $\delta>0$), $x\in X$.

Hence, by continuity of f, $d(f(x),f(\bar{x}))<\varepsilon$ and $f(\bar{x})\in f(E)$. $f(E)$ must be dense in $f(X)$.

~~b)~~

b) $\because$ ~~X dense in~~ E dense in X, $\therefore \forall x\in X$, $\exists \{x_n\}\subset E$ s.t. $x_n\to x$.

f is continuous $\Rightarrow f(x_n)\to f(x)$. Similarly, $g(\cancel{x_n}\ y_n)\to g(x)$. (Say, y_n is a sequence in E as well).

We know $\forall x\in E X$, $f(x)=g(x)$. This means x_n, y_n converges to the same limit point ~~in~~ of E.

Thus $f=g$ following the definition of continuity.

c) $f(x):=\begin{cases} x & x\in\mathbb{R}\setminus\mathbb{Q} \\ 0 & x\in\mathbb{Q}\end{cases}$

pick $x\in\mathbb{R}\setminus\mathbb{Q}$, $x\in(c,c-\delta)$ for some small, positive δ. let $c\in\mathbb{Q}$. We can do this because ~~both rational~~ irrational numbers are dense in $\mathbb{R}$.

Thus, we have $|x-c|<\delta$, but $|f(x)-f(c)|=x>$ (some small, arbitrary ε).

↳ This function is not continuous.

7(a) follows similar logic.

let ~~$x\in E$~~, an irrational $x\in(c,c-\delta)$ for some $\delta>0$, $c\in\mathbb{Q}$.

when $|x-c|<\delta$, $|f(x)-f(c)|=x^2-1>\varepsilon$ (when $x>1$ or $x<-1$).

↳ This function is not continuous.

4.

a) This limit does not exist. $\lim_{x\to0^+}\frac{\cos x}{x}=+\infty$, while $\lim_{x\to0^-}\frac{\cos x}{x}=-\infty$.

b) $\lim_{x\to0}\frac{\sqrt{1+3x+x^2}-1}{x}$ = ~~$\lim_{x\to0}\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}-\lim_{x\to0}\frac{1}{x}$~~ $\lim_{x\to0}\frac{x+3}{\sqrt{1+3x+x^2}+1}=\frac{3}{1+1}=\frac{3}{2}$

c) $\lim_{x\to0}\frac{1-\cos x}{x^2}$ ~~$=\frac{2\sin^2(\frac{x}{2})}{x}=\left(\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\right)^2$~~

$=\lim_{x\to0}\frac{2\sin^2(\frac{x}{2})}{x}=\lim_{x\to0}\left(\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\right)^2\cdot\frac{x}{2}=1\cdot0=0$.

7.

a) $x\in\{1\}\cup\{-1\}$

b) $x\in$ ~~$\mathbb{R}\setminus\{x\}$~~ $[0,1)$

10.

$Z(f)=f^{-1}(\{0\})$, but $\{0\}$ is closed using usual topology.

Since f is continuous, $Z(f)=f^{-1}(\{0\})$ must be closed.

(i.e., $f(x_n)\to f(x)$, but $f(x_n)=0\ \forall n$, thus $x\in Z(f)$)

8.

$\lim_{h\to0}(f(x+h)-f(x-h))=\lim_{h\to0}(f(x+h)-f(x))+\lim_{h\to0}(f(x)-f(x-h))$ ~~$=0$~~.

$\because f$ is continuous, $\therefore d(f(x+h),f(x))$ or $d(f(x),f(x-h))<\varepsilon$ $\forall\varepsilon>0$ (arbitrarily small) fixing $h>0$ ~~$\forall h>0$~~ $\Rightarrow$ previous limits $=0$.

Hence $\lim_{h\to0}(f(x+h)-f(x-h))=0+0=0$.