Homework Assignment Solution 7

October 22, 2021

Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Show that f is continuous if and only if $f^{-1}((-\infty, c))$ and $f^{-1}((c, \infty))$ are open for every $c \in \mathbb{R}$.

(b) Deduce that the same equivalence holds with the intervals "$(-\infty, c)$," and "(c, ∞)" replaced by "$(-\infty, c]$," and "$[c, \infty)$," respectively, and "open" replaced by "closed."

(c) Deduce that f is continuous implies $f^{-1}(\{c\})$ is closed for every $c \in \mathbb{R}$. Show that the opposite implication is false.

(a) Since the open rays open, by Theorem 7.8, the "only if" direction is evident. So suppose that $f : \mathbb{R} \to \mathbb{R}$ is such that the preimage of every open ray is open.

Let $\epsilon > 0$ be given, and let $a \in \mathbb{R}$ be fixed. We will show that f is continuous at $x = a$ by showing that there exists $\delta > 0$ such that for every x satisfying $|x - a| < \delta$, $|f(x) - f(a)| < \epsilon$.

By the assumption, $f^{-1}((-\infty, f(a)+\epsilon))$ and $f^{-1}((f(a)-\epsilon, \infty))$ are open. Then the intersection

$$A := f^{-1}((-\infty, f(a)+\epsilon)) \cap f^{-1}((f(a)-\epsilon, \infty))$$

is also open by Theorem 3.12.(2). Observe that this is precisely the set

$$A = \{x \in \mathbb{R} : f(a) - \epsilon < f(x) < f(a) + \epsilon\} = \{x \in \mathbb{R} : |f(x) - f(a)| < \epsilon\}.$$

Since $f(a) - f(a) = 0$, clearly $a \in A$. Since A is open, a is an interior point, hence there exists some δ-neighborhood of a, which is the open interval $(a - \delta, a + \delta)$, such that $(a - \delta, a + \delta) \subseteq A$. But since every $x \in A$ satisfies the property that $|f(x) - f(a)| < \epsilon$, the assertion has been proved.

(b) By Corollary 7.9, we have that f is continuous if and only if the preimage of every closed set is closed. Hence again, the "only if" direction is obvious (as the rays $(-\infty, c] = (c, \infty)^c$ and $[c, \infty) = (-\infty, c)^c$ are closed).

If $f^{-1}((-\infty,c])$ and $f^{-1}([c,\infty))$ are closed for every $c\in\mathbb{R}$ then

$$f^{-1}((-\infty,c)) = f^{-1}([c,\infty)^c) = (f^{-1}([c,\infty)))^c,$$

and

$$f^{-1}((c,\infty)) = f^{-1}((-\infty,c]^c) = (f^{-1}((-\infty,c]))^c$$

are open (by Thm. 3.10), where we also used question 5(b) below. Thus f is continuous by (a).

(c) Note that $f^{-1}(\{c\}) = f^{-1}((-\infty,c])\cap f^{-1}([c,\infty))$ (by question 5(c) below), which is closed by (b). (Alternatively, one could use the fact that $\{c\}$ is closed in conjunction with Corollary 7.9).

As for the remaining part, consider $f(x) := x-[x]$. Then

$$f^{-1}(\{c\}) = \begin{cases} \varnothing & \text{if } c<0 \text{ or } c\geq 1 \\ \{n+c : n\in\mathbb{N}\} & \text{if } c\in[0,1). \end{cases}$$

In either case it is closed, but f is not continuous.

Problem 2

For each of the following conditions, give an example of $f:\mathbb{R}\to\mathbb{R}$ that satisfies it:

(a) $(\forall x)(\forall y)(\forall\epsilon>0)(\forall\delta>0)\ |x-y|\leq\delta \implies |f(x)-f(y)|\leq\epsilon$;

(b) $(\forall x)(\forall\epsilon>0)(\exists\delta>0)$ such that $\forall y\ |x-y|\leq\epsilon \implies |f(x)-f(y)|\leq\delta$;

(c) $(\forall x)(\forall\epsilon>0)(\exists\delta>0)$ such that $\forall y\ |f(x)-f(y)|>\epsilon \implies |x-y|>\delta$;

(d) $(\forall x)(\exists\epsilon>0)$ such that $(\forall\delta>0)(\forall y)\ |x-y|\leq\delta \implies |f(x)-f(y)|\leq\epsilon$, but f is discontinuous;

(e) f is continuous, but it is not true that $(\forall x)(\forall\epsilon>0)(\exists\delta>0)$ such that $\forall y\ x-y\leq\delta \implies |f(x)-f(y)|\leq\epsilon$;

(f) Same as (e), but with "f is continuous" replaced by $(\forall x)(\forall\epsilon>0)(\exists\delta>0)$ such that $\forall y\ x-y\leq\delta \implies f(x)-f(y)\leq\epsilon$.

The plots are given on the following page.

(a) This means that for any pair x,y, $|f(x)-f(y)|$ is less than any positive number, i.e., $f(x)=f(y)$ for every x,y. A constant function is the only function that has that property.

(b) This means that for each x and $\epsilon>0$ given, there exists $\delta>0$ corresponding to that x and ϵ such that whenever y satisfies $|x-y|\leq\epsilon$, $|f(x)-f(y)|\leq\delta$. Therefore any function that does not diverge to $+\infty$ or $-\infty$ in finite time[1] will do, such as $y=x$.

[1] Diverging in finite time means that there exists some $-\infty<a<\infty$ such that $\lim_{x\to a} f(x) = \pm\infty$, such as $1/x$ or $1/(1-x)$ or $\tan x$.

(c) This means that for each x and $\epsilon > 0$ given, there exists $\delta > 0$ corresponding to that x and ϵ such that whenever y satisfies $|f(x) - f(y)| > \epsilon$, $|x - y| > \delta$. Taking $y = x$, we can take $\delta = \epsilon$.

(d) This means that for each x, there exists some $\epsilon > 0$ such that for every y, $|f(x) - f(y)| \leq \epsilon$, i.e., $f(x)$ must be bounded. Yet it has to be discontinuous, so the step function $f(x) = 1$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$ will do.

(e) The "not true" condition means that for each x and $\epsilon > 0$, there exists $\delta > 0$ corresponding to that x and ϵ such that whenever y satisfies $x - y \leq \delta$, $|f(x) - f(y)| \leq \epsilon$. If we take $y = -x$, and $\epsilon = 1$, $x = 0$, we see that $|f(x) - f(y)| = |y|$. Yet since $x - y = -y \leq 0$, no matter what δ is chosen, any y will satisfy the condition $x - y \leq \delta$. But choosing $y > 1$, we see that $|f(x) - f(y)| = |y| > 1$.

(f) Any function f such that for each x and $\epsilon > 0$, there exists $\delta > 0$ such that for every y satisfying $x - y \leq \delta$ implies $f(x) - f(y) \leq \epsilon$ must be an increasing function, because if there exists a pair where $y > x$ and $f(y) < f(x)$, then by choosing $\epsilon < f(x) - f(y)$, we get a contradiction (since $x - y < 0 < \delta$ for any $\delta > 0$). So use $y = x$.

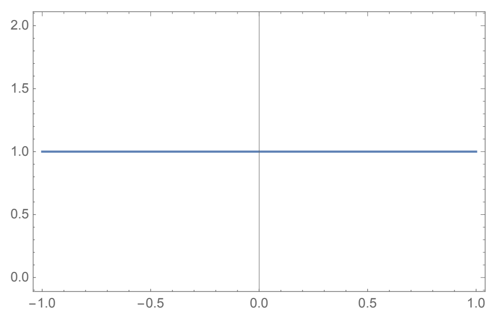

Figure 1: Plot for (a)

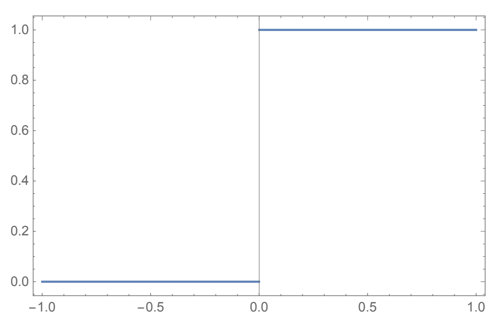

Figure 2: Plot for (d)

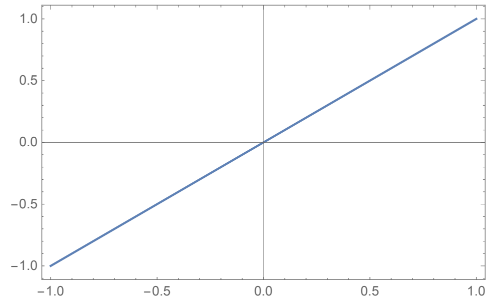

Figure 3: Plot for (b), (c), and (f)

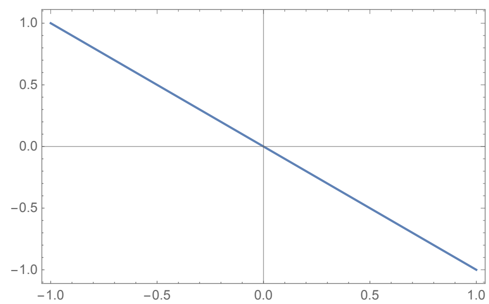

Figure 4: Plot for (e)

Problem 3

Let $f : X \to Y$ be continuous, and let $E \subset X$ be dense in X.

(a) Show that $f(E)$ is dense in $f(X)$.

(b) Deduce that if $f, g : X \to Y$ are two continuous fuctions such that $f(x) = g(x)$ for every $x \in E$, then $f = g$.

(c) Use (b) to deduce that the function

$$f(x) := \begin{cases} x & x \text{ is irrational,} \\ 0 & x \text{ is rational,} \end{cases}$$

and the function in question 7(a) below are not continuous.

(a) If $f(E) = f(X)$, then there is nothing to be done. Assume $f(x) \in f(X) \setminus f(E)$. We will show that $f(x)$ is a limit point of $f(E)$. To do so, we will find a sequence $(e_n)_{n=1}^{\infty} \subseteq E$ such that $f(e_n) \neq f(x)$ for every n, and $f(e_n) \to f(x)$.

Because E is dense in X, there exists a sequence $(e_n)_{n=1}^{\infty} \subseteq E$, $e_n \neq x$ for every n, and $e_n \to x$. (Else $x \in E$, which is forbidden here.) By Theorem 7.2, we have $\lim_{n\to\infty} f(e_n) = f(x)$.

(b) Suppose f and g are functions that agree on a dense subset. For every $z \in X \setminus E$ (which must be a limit point of E), there exists a sequence $(z_n)_{n=1}^{\infty} \subseteq E$

such that $z_n \neq z$ for every n and $z_n \to z$. Since f and g agree on E we have that $f(z_n) = g(z_n)$, and so, by continuity,

$$f(z) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = g(z).$$

(c) Since the set of rational numbers is dense in $\mathbb{R}$, and since $f(x)$ agrees with the constant function $g(x) = 0$ on $\mathbb{Q}$, by (b), if f were continuous, f must also be the constant function zero, which it clearly is not: $f(\sqrt{2}) = \sqrt{2} \neq 0$. For 7(a), by the same reasoning, $f(\sqrt{2}) = 1 \neq 0$.

Problem 4

Find the limit, or prove that it does not exist;

(a) $\lim_{x\to 0} \frac{1}{x}\cos x$;

(b) $\lim_{x\to 0} \frac{\sqrt{1+3x+x^2}-1}{x}$;

(c) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

(a) We claim that the limit does not exist. Consider two sequences $a_n := (1/2n\pi)_{n=1}^{\infty}$ and $b_n := (-1/2n\pi)_{n=1}^{\infty}$, both of which converging to 0. Then

$$\lim_{n\to\infty} \frac{1}{a_n}\cos a_n = \lim_{n\to\infty} 2n\pi \neq \lim_{n\to\infty}(-2n\pi) = \lim_{n\to\infty} \frac{1}{b_n}\cos b_n,$$

which violates Theorem 7.2.

(b) Note that

$$\frac{\sqrt{1+3x+x^2}-1}{x} = \frac{3x+x^2}{x\sqrt{1+3x+x^2}+x} = \frac{3+x}{\sqrt{1+3x+x^2}+1}.$$

We show now that $f(x) = \sqrt{x}$ is a continuous at $x = 1$. Indeed, for $\epsilon > 0$ given,

$$\left|\sqrt{x}-1\right| = \left|\frac{x-1}{\sqrt{x}+1}\right| < |x-1| < \epsilon,$$

provided that we choose $\delta = \epsilon$. Hence by Corollary 7.6 and Theorem 7.7, the function $\frac{3+x}{\sqrt{1+3x+x^2}+1}$ is continuous. Therefore, the limit equals to the value at $x = 0$, which is $3/2$.

(c) Observe that

$$\frac{1-\cos x}{x^2} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x},$$

which converges to $1/2$ (by Cor. 7.4, as sin and cos are continuous and $(\sin x)/x \to 1$ as $x \to 0$ by the comment).

Extra: In the calculation below we give an elementary proof of the facts that sin is a continuous function and that $(\sin x)/x \to 1$ as $x \to 0$.

From the definition of $\sin x$ and $\cos x$, we get the following inequality (see the figure below):

For $0 < x < \pi/2$, we have

$$x \cos x < \sin x < x \implies \cos x < \frac{\sin x}{x} < 1,$$

and for $0 > x > -\pi/2$, we have

$$x < \sin x < x \cos x \implies \cos x < \frac{\sin x}{x} < 1.$$

So on the interval $(-\pi/2, \pi/2)$, we have

$$\cos x < \frac{\sin x}{x} < 1.$$

We will use this inequality later. To show that $\sin x$ is continuous, it suffices to use the fact that $|\sin x| < |x|$. Using this and one of the trigonometric identities $\sin x - \sin y = 2\cos((x+y)/2)\sin((x-y)/2)$, we can show that $\sin x$ is continuous at $x = a$:

$$|\sin x - \sin a| = \left|2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right| \leq 2\left|\sin\frac{x-a}{2}\right| \leq |x-a| < \epsilon$$

upon choosing $\delta = \epsilon$. Hence $\cos x = \sin(x + \pi/2)$ too, is continuous. This shows that $\lim_{x\to 0} 1/(1+\cos x) = 1/2$ by choosing any sequence (x_n) converging to 0.

To show that $\sin x/x \to 1$, use the inequality we found previously in conjunction with the fact that $\cos x$ is continuous:

$$\cos x < \frac{\sin x}{x} < 1 \implies \lim_{n\to\infty} \cos\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin 1/n}{1/n} \leq 1 \implies \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

by the squeeze theorem and Theorem 7.2. Hence by Corollary 7.4, the limit is 1/2.

Problem 5

Let $f : X \to Y$ be any function. Show the following properties of the preimage:

(a) $f^{-1}(\bigcup_{\alpha\in I} V_\alpha) = \bigcup_{\alpha\in I} f^{-1}(V_\alpha)$ for any union $\bigcup_{\alpha\in I} V_\alpha \subseteq Y$ of subsets of Y;

(b) $f^{-1}(U^c) = (f^{-1}(U))^c$ for any $U \subseteq Y$;

(c) Deduce from (b) that (a) also holds with the union "$\bigcup_{\alpha\in I}$" replaced by the intersection "$\bigcap_{\alpha\in I}$."

(a) Note that

$$\begin{aligned}
x \in f^{-1}(\bigcup_{\alpha\in I} V_\alpha) &\Leftrightarrow f(x) \in \bigcup_{\alpha\in I} V_\alpha \\
&\Leftrightarrow \exists_{\alpha\in I} \text{ s.t. } f(x) \in V_\alpha \\
&\Leftrightarrow \exists_{\alpha\in I} \text{ s.t. } x \in f^{-1}(V_\alpha) \\
&\Leftrightarrow x \in \bigcup_{\alpha\in I} f^{-1}(V_\alpha)
\end{aligned}$$

(b) Note that

$$\begin{aligned}
x \in f^{-1}(U^c) &\Leftrightarrow f(x) \in U^c \\
&\Leftrightarrow f(x) \notin U \\
&\Leftrightarrow \text{ it is not true that } x \in f^{-1}(U) \\
&\Leftrightarrow x \in (f^{-1}(U))^c
\end{aligned}$$

(c) By De Morgan law,

$$\begin{aligned}
\left(f^{-1}\left(\bigcap_{\alpha\in I} V_\alpha\right)\right)^c &= f^{-1}\left(\left(\bigcap_{\alpha\in I} V_\alpha\right)^c\right) = f^{-1}\left(\bigcup_{\alpha\in I} V_\alpha^c\right) \\
&= \bigcup_{\alpha\in I} f^{-1}(V_\alpha^c) = \bigcup_{\alpha\in I} \left(f^{-1}(V_\alpha)\right)^c \\
&= \left(\bigcap_{\alpha\in I} f^{-1}(V_\alpha)\right)^c,
\end{aligned}$$

where we used (b) in the first and fourth equalities, and (a) in the third equality. Taking complements proves the claim.

Problem 6

Show that the definition of continuity (Definition 7.5) is equivalent to the one where the "$\le$" inequalities are replaced by the "$<$" inequalities. Namely, show that, given $f : X \to Y$ is such that

$$(\forall \epsilon > 0)(\exists \delta > 0) \quad d_Y(f(x), f(y)) \le \epsilon \quad \text{if } d_X(x, y) \le \delta,$$

then also

$$(\forall \epsilon > 0)(\exists \delta > 0) \quad d_Y(f(x), f(y)) < \epsilon \quad \text{if } d_X(x, y) < \delta,$$

and vice versa.

We first show that $\le \implies <$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), f(y)) \le \epsilon/2$ if $d_X(x, y) \le \delta$, so for this δ, $d_Y(f(x), f(y)) < \epsilon$.

Now we show $< \implies \le$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that whenever $d_X(x, y) < \delta$, $d_Y(f(x), f(y)) < \epsilon$. Choose $\delta/2$ instead. Then whenever $d_X(x, y) \le \delta/2$, we sure have $d_X(x, y) < \delta$, therefore $d_X(f(x), f(y)) < \epsilon \le \epsilon$.

Problem 7

Find the points $x \in \mathbb{R}$ at which f is continuous.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \begin{cases} x^2 - 1 & x \text{ is irrational,} \\ 0 & x \text{ is rational;} \end{cases}$

(b) $f : [0,1] \to \mathbb{R}$, $f(x) := \sup\{1 - x^n : n \in \mathbb{N}\}$.

(a) We will show first that $f(x)$ is continuous at $x = \pm 1$, and then show that at any other point, it is discontinuous. Let $(x_n)_{n=1}^{\infty}$ be any sequence of reals converging to 1. If $x_n \in \mathbb{Q}$, then $f(x_n) = 0$ for that n. If $x_n \in \mathbb{R} \setminus \mathbb{Q}$, then for N sufficiently large, for every $n > N$ for which x_n is irrational, we can ensure $0.5 < x_n < 1.5$, and so

$$|f(x_n)| = |x_n^2 - 1| = |x_n - 1||x_n + 1| \leq 2.5|x_n - 1|.$$

Choose N sufficiently large (large than what we had to choose to ensure that $0.5 < x_n < 1.5$) so that for all $n > N$, $|x_n - 1| < \epsilon/2.5$. Hence $f(x_n) \to 0$ for any sequence (x_n) converging to 1, hence f is continuous at $x = 1$. Exactly the same argument can be used for $x = -1$.

Let $x = a \neq \pm 1$. Assume further that it is irrational. Then $f(a) \neq 0$. By PS4 #1, there exists a rational sequence (q_n) converging to a, hence if f were continuous at $x = a$, then we need to have $\lim_{n\to\infty} f(q_n) = f(a) \neq 0$, yet $f(q_n) \equiv 0$ for all n, a contradiction. Hence $f(x)$ cannot be continuous at any irrational number.

Let now $x = a \neq \pm 1$ be rational. Then $f(a) = 0$. Take $x_n = a + \sqrt{2}/n$. Then clearly $x_n \to a$, but since $a + \sqrt{2}/n$ is always irrational,

$$f(x_n) = \left(a + \frac{\sqrt{2}}{n}\right)^2 - 1 = a^2 + \frac{2\sqrt{2}a}{n} + \frac{2}{n^2} - 1 \to a^2 - 1 \neq 0,$$

as $a \neq \pm 1$.

(b) Note that for any $x \in [0,1)$, $1 - x^n < 1 - x^m$ if $n < m$. Moreover, since $a^n \to 0$ as $n \to \infty$ when $|a| < 1$, $\sup\{1 - x^n : n \in \mathbb{N}\} = 1$ if $x \in [0,1)$, and $f(x) = 0$ if $x = 1$.

We show that $f(x)$ is continuous on $[0,1)$. Let $\epsilon > 0$ be given and let $a \in [0,1)$ be fixed. Then for all $x \in N_{\frac{1-a}{2}}(a) \cap [0,1]$, $f(x) = 1$.

At $x = 1$, $f(x)$ fails to be continuous since if $\epsilon = 1/2$, no $\delta > 0$ can ensure that whenever $1 - \delta < x \leq 1$, $|f(x)| < \epsilon$.

Problem 8

Let $f : \mathbb{R} \to \mathbb{R}$, and fix $x \in \mathbb{R}$. Show that if f is continuous at x, then

$$\lim_{h\to 0}(f(x+h) - f(x-h)) = 0,$$

but that the opposite implication is false.

Let $\epsilon > 0$ be given. Continuity of f at x implies that there exists $\delta > 0$ such that whenever y is such that $|x-y| < \delta$, $|f(x) - f(y)| < \epsilon/2$. Then whenever $|h| < \delta$,

$$\begin{aligned} |f(x+h) - f(x-h)| &= |f(x+h) - f(x) + f(x) - f(x-h)| \\ &\le |f(x+h) - f(x)| + |f(x) - f(x-h)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Take $f(x) := x$ whenever $x \neq 0$, and $f(0) = 1$. Then $\lim_{x\to 0} f(x) = 0$, yet $f(0) = 1$, so $f(x)$ is not continuous at $x = 0$. However

$$\lim_{h\to 0}(f(h) - f(-h)) = \lim_{h\to 0}(2h) = 0.$$

Problem 9

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $|f|$, $\max\{f, g\}$, and $\min\{f, g\}$ are continuous.

We show that $|f|$ is continuous. Let $\epsilon > 0$ be given. Since f is continuous, there exists δ such that whenever y satisfies $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Then

$$||f(x)| - |f(y)|| < |f(x) - f(y)| < \epsilon$$

by PS2 #7. Since

$$\max\{f, g\} = \frac{|x - y| + x + y}{2},$$

which is the composition, sum, and constant multiple of continuous functions, $\max\{f, g\}$ is continuous, and similarly for $\min\{f, g\}$.

Problem 10

Let $f : X \to Y$ be continuous, and let $Z(f) := \{x \in X : f(x) = 0\}$ denote the *zero set* of f. Show that $Z(f)$ is closed.

Note that $Z(f) = f^{-1}(\{0\})$. By #1 (c), $f^{-1}(\{0\})$ is closed.