Math 425A Problem Sheet 8 (due 9am on Monday, 25th Oct)

Essential problems

1. (4 pts) Let $V \subset \mathbb{R}$ be a set that is not compact. Show that

(a) there exists a continuous function defined on V that is not bounded,

(b) there exists a continuous and bounded function defined on V which does not attain its supremum on V.

(Comment: this question shows that the assumption of compactness in Lem. 8.4 and in Thm. 8.5 is necessary. In fact, it shows more: that in the case of subsets of $\mathbb{R}$, for ***any*** *noncompact set we can find a counterexample; also hint: you may want to consider the cases when V is bounded/unbounded and functions of the form $1/(1-(x-x_0)^2)$, $1/(x-x_0)$, $x^4/(1+x^4)$. Also note that if V is bounded then there exists a limit point of V that does not belong to V.)*

2. (3 pts) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x=0$, $f(0)=0$ and that f is subadditive, i.e. that

$$f(x+y) \leq f(x) + f(y).$$

Show that f is uniformly continuous.

3. (3 pts) Suppose that $f\colon (a,b) \to \mathbb{R}$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ is continuous and that $\lim_{x\to b-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist (and are finite). Show that then f is uniformly continuous. *(Hint: Use Thm. 8.11.)*

Additional problems

4. (1 pt) Give an example of a bounded function on $[0,1]$ that attains neither its infiumum nor maximum. Why does it not contradict Thm. 8.5?

5. (1 pt) Let $n \in \mathbb{N}$ and let

$$P(x) := x^{2n} + a_{2n-1}x^{2n-1} + \ldots a_1 x + a_0$$

for some $a_0, \ldots, a_{2n-1} \in \mathbb{R}$. Show that P attains its infimum on $\mathbb{R}$. *(Comment: This shows that one of the claims of Thm. 8.5 applies also to some continuous functions defined on* ***noncompact*** *domains.)*

6. (1 pt) Let $f := \mathbb{R} \to \mathbb{R}$ be continuous and periodic (i.e. that there exists $c > 0$ such that $f(x) = f(x+c)$ for all $x \in \mathbb{R}$). Show that for any such function the claims of Thm. 8.2, Lem. 8.4, Thm. 8.5 and Thm. 8.11 remain valid; namely that the range of f is compact, that f is bounded and attains its bounds, and is uniformly continuous.

7. (1 pt) For each of the following functions determine whether it uniformly continuous. If not, is it continuous?

(a) $f\colon [1,\infty) \to \mathbb{R}$, $f(x) := \sin x$,

(b) $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) := |x|$,

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \arctan x$,

(d) $f\colon (0,\infty) \to \mathbb{R}$, $f(x) := \cos x \cdot \cos \frac{\pi}{x}$,

(e) $f\colon (0,1) \to \mathbb{R}$, $f(x) := \mathrm{e}^{-\frac{1}{x}}$,

(Hint: you might want to use questions 3 and 6 above and Lem. 8.13; you can also use the fact that all trigonometric functions are continuous on their domains.)

8. (1 pt) Give an example of a noncompact subset U of $\mathbb{R}$ such that every continuous $f\colon U \to \mathbb{R}$ is uniformly continuous.

9. (1 pt) Let $f\colon X \to Y$. Show that f is continuous if an only if f is uniformly continuous on every compact subset $K \subset X$. *(Hint: Recall PS3.10.)*

10. (1 pt) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $(a,b]$ and on $[b,c)$, then it is uniformly continuous on (a,c).

11. (1 pt) Give an example of two closed sets $A, B \subset \mathbb{R}$, and $f\colon A \cup B \to \mathbb{R}$ such that f is uniformly continuous on A and on B, but it is not uniformly continuous on $A \cup B$.

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