MATH 425a PS8

1. a) let $V := [0,\infty) \subset \mathbb{R}$ and $f := x$. hence $f(x)$ is continuous but unbounded.

b) let $V := [0,\infty) \subset \mathbb{R}$ and $f := \arctan x$, hence $|f(x)| < \frac{\pi}{2}$ (which means $f(x)$ is bounded) but it does not attain its supremum although it's continuous.

2. ~~Given f continuous at $x=0$ $f(0)=0$, $\Rightarrow \forall \varepsilon>0$, $\forall x$ $\exists \delta>0$ s.t. $f(x)<\varepsilon$ whenever $x<\delta$.~~

~~Similarly, $\forall \varepsilon>0$, $\forall y$ $\exists \delta>0$ s.t. $f(y)<\varepsilon$ whenever $y<\delta$.~~

We want to show $\forall \varepsilon>0, \exists \delta>0$ ~~$\forall x$ s.t. $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$.~~

Given f is continuous at $x=0$, $f(0)=0$, $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(0)|<\varepsilon$ whenever $|x-0|<\delta$.

$\hookrightarrow |f(x)|<\varepsilon$ whenever $|x|<\delta$.

~~Hence when~~ WLOG Let's assume $x>y$, so that $x \neq y$. From f's ~~the~~ continuity at 0 ~~$x=0$~~,

$\hookrightarrow$ when $x-y<\delta$, $f(x-y)<\varepsilon$ (think about an invisible $f(0)$, $x-y \in \mathbb{R}$).

We also know ~~$f(x-y) \le f(x)+f(-y)$~~ $f(x) = f(x-y+y) \le f(x-y)+f(y)$

due to its subadditive property. Hence $f(x)-f(y) \le f(x-y)$.

Combine this with $f(x-y)<\varepsilon$, $\Rightarrow f(x)-f(y)<\varepsilon$ whenever $x-y<\delta$.

$\forall \varepsilon>0$, This proves that f is uniformly continuous.

(Absolute sign here eliminated WLOG).

3. $\lim_{x\to a^+} f(x)$ exists $\Rightarrow \exists a' \in (a,b)$ ~~$a'>a$~~ s.t. $|f(t)-\lim_{x\to a^+} f(x)|<\frac{\varepsilon}{2}$ $\forall t \in (a,a')$ (i)

$\lim_{x\to b^-} f(x)$ exists $\Rightarrow \exists b' \in (a,b)$ s.t. $|f(t)-\lim_{x\to b^-} f(x)|<\frac{\varepsilon}{2}$ $\forall t \in (b',b)$. (ii)

We define $(a,b) = (a,a') \cup [a',b'] \cup (b',b)$. $\Rightarrow [a',b']$ compact.

$\because f$ is continuous, $\therefore f$ is uniformly continuous on $[a',b']$.

$\Rightarrow \forall \frac{\varepsilon}{2}>0, \exists \delta'$ s.t. $|f(x)-f(y)|<\frac{\varepsilon}{2}$ if $|x-y|<\delta'$ $\forall x,y \in [a',b']$.

let $\delta = \min(\delta', |a-a'|, |b-b'|)$. Thus, $\forall x,y \in (a,b)$, $|x-y|<\delta$:

① x,y in the same ~~interval~~ subinterval:

(1) $x,y \in [a',b']$: U.C. already discussed

(2) $x,y \in (a,a')$ or (b',b): $|f(x)-f(y)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ (from (i))

② x,y in different subintervals

(1) one of them $\in [a',b']$: WLOG assume $x \in (a,a')$, $b \in [a',b']$.

Then given $|x-y|<\delta$, $|f(x)-f(y)| \le |f(x)-f(a')| + |f(a')-f(y)|$

$$< \underbrace{\varepsilon}_{\text{by (i)}} + \underbrace{\frac{\varepsilon}{2}}_{\text{U.C. on } [a',b']} = \frac{3}{2}\varepsilon.$$

since ε arbitrarily small, $|f(x)-f(y)|<\varepsilon$.

(2) neither in $[a',b']$: WLOG assume $x \in (a,a')$, $b \in (b',b)$.

By a similar process we conclude $|f(x)-f(y)|<\varepsilon$.

Thus from ①②, f is uniformly continuous on (a,b).

4. Think about ~~$f(x):=$ $y=(x-0.5)^2$ $y=(x-1)^2$~~ $f(x):=\begin{cases}(x-1)^2 & x\in[0,1]\setminus\{0\}\cup\{1\}\\ 0.5 & x=0 \text{ or } 1\end{cases}$

It does not contradict theorem 8-5 because it's not continuous.

8. let $U:=\mathbb{Z}^+$. This is a non-compact, discrete set, so a continuous $f: U\to\mathbb{R}$ uniformly continuous.

11. let $A:=\mathbb{Z}^+$, $B:=\{n+\frac{1}{n} \mid n\in\mathbb{Z}^+\}$. Both A, B are closed and discrete.

Define $f: A\cup B\to\mathbb{R}$ ~~=~~ $f(x):=\begin{cases}0 & x\in A\\ 1 & x\in B\end{cases}$.

10. $\because f$ uniformly continuous on $[a,b]$ $\therefore \exists\delta_1$ s.t. $\forall\varepsilon>0$ $|x-y|<\delta_1 \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}$ $\forall x,y\in[a,b]$

Similarly, f uniformly continuous on $[b,c] \Rightarrow \exists\delta_2$ s.t. $\forall\varepsilon>0$, $|x-y|<\delta_2 \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}$ $\forall x,y\in[b,c)$

let $\delta=\frac{1}{2}\min\{\delta_1,\delta_2\}$.

① $x,y\in[a,b] \Rightarrow$ obvious that f uniformly continuous.

② $x,y\in[b,c) \Rightarrow$ same as above.

③ WLOG assume $x\in$ ~~$[a,b)$~~ $(a,b]$, $y\in[b,c) \Rightarrow$ when $|x-y|\le|x-b|+|b-y|<\delta$, ~~$x<k<y$, $k\in\mathbb{R}$, $k\in[a,c]$~~

$\hookrightarrow |f(x)-f(y)|\le|f(x)-f(b)|+|f(b)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$ (from uniform continuity of f, also $|x-b|, |b-y| <\frac{\delta}{2}$)

$=\varepsilon$ $\forall\varepsilon>0$.

$\Rightarrow f$ is uniformly continuous on (a,c) from ①②③