Solutions to Problem Set 8

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1. (a) Since $V \subset \mathbb{R}$, we can deduce that being non compact is equivalent to either V being not bounded or not closed (due to Heine-Borel, Thm 4.8). We shall construct a function for each of these two cases.

   - If V is not bounded, then clearly the function $f : V \to \mathbb{R}$ defined as $f(x) := x$ works.

   - If V is not closed, then there exists a limit point of V that is not in V. Let a be this limit point. Define $f : V \to \mathbb{R}$ as $f(x) := \frac{1}{x-a}$. This is clearly continuous (as in Ex. 7.12) as a is not in the domain V. Now we prove that it is unbounded. Let $M > 0$ be arbitrary. Since a is a limit point of V, there exists a sequence $(v_n) \subset V$ such that $v_n \to a$. Therefore, there exists N such that $|v_N - a| < \frac{1}{M}$ which implies $|f(v_N)| > M$. As M was arbitrary this proves f is unbounded.

   (b) We again break the problem into two cases as before

   - If V is not bounded consider $f : V \to \mathbb{R}$ defined as $f(x) := \frac{x^4}{1+x^4}$. Clearly $0 < f(x) < 1$ for all $x \in V$ and f is continuous. Since V is unbounded, there exists a sequence (v_n) in V such that $|v_n| \to \infty$. Then one can compute that $\lim_{n\to\infty} f(v_n) = 1$, which shows that given any $\varepsilon > 0$, there exists N such that $f(v_N) > 1 - \varepsilon$. Therefore $1 - \varepsilon$ is not an upper bound of f for any $\varepsilon > 0$ and we have seen that f is bounded by 1 proving that is indeed the supremum. The supremum is not attained because the equation $\frac{x^4}{1+x^4} = 1$ has no real solutions.

   - If V is not closed, assume as before that a be a limit point of V not in V. Define $f : V \to \mathbb{R}$ as $f(x) := \frac{1}{1+(x-a)^2}$. Now we follow an argument almost identical to the previous case. Observe again that $0 < f(x) < 1$. Since a is a limit point of V, there exists a sequence (v_n) in V such that $v_n \to a$. Then one can compute that $\lim_{n\to\infty} f(v_n) = 1$, which shows that given any $\varepsilon > 0$, there exists N such that $f(v_N) > 1 - \varepsilon$. Therefore $1 - \varepsilon$ is not an upper bound of f for any $\varepsilon > 0$ and we have seen that f is bounded by 1 proving that is indeed the supremum. The supremum is not attained because the equation $\frac{1}{1+(x-a)^2} = 1$ has the only solution $x = a$ which is not in the domain (by assumption).

2. Let $\varepsilon > 0$. By the continuity of f at 0 we have that there exists a $\delta > 0$ such that whenever $|x| < \delta$ we have $|f(x) - f(0)| < \varepsilon$. $f(0) = 0$ is also given, so the above line reduces to saying that whenever $|x| < \delta$ we have $|f(x)| < \varepsilon$.

   Thus for any x, y with $|x - y| \leq \delta$ we have, from subadditivity,

   $$f(x) - f(y) \leq f(x - y) \leq |f(x - y)| \leq \varepsilon,$$

   and similarly (by the same argument, but with x, y switched) $f(y) - f(x) \leq f(y - x) \leq |f(y - x)| \leq \varepsilon$. Thus $|f(x) - f(y)| \leq \varepsilon$, proving uniform continuity.

3. Let us break this problem into three few cases depending upon whether a, b are finite or not.

   - Consider first $a, b \in \mathbb{R}$ i.e, both finite. Then by the given conditions, f can be extended continuously to the closed interval $[a, b]$ (by letting $f(a) := \lim_{x \to a^+} f(x)$, $f(b) := \lim_{x \to b^-} f(x)$). However since $[a, b]$ is compact f is automatically uniformly continuous on that (using Thm 8.11) and that implies that f is uniformly continuous on (a, b) too (as if a function is uniformly continuous on a set it is uniformly continuous over any subset of that set too).

   - Now let's assume $a \in \mathbb{R}$ and $b = +\infty$. We are therefore given that $\lim\limits_{x \to \infty} f(x)$ is finite. Assume that the limit is equal to L. Fix some $\varepsilon > 0$. Choose M such that $x > M$ implies $|f(x) - L| < \frac{\varepsilon}{2}$. Consider the compact interval $[a, M + 1]$. Since this is compact, f is uniformly continuous on it and we can find $\delta > 0$ such that for any $x \in [a, M + 1]$ whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

     Next, observe that we can choose so that $\delta < 1$ (because if the uniform continuity condition holds for some choice of δ it holds for all smaller values too). Now if $x \in (M + 1, \infty)$ we can see that any y satisfying $|x - y| < \delta$ must be in (M, ∞) (this was the whole point of choosing $\delta < 1$). However that implies,

     $$|f(x) - f(y)| < |f(x) - L| + |f(y) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

     from the triangle inequality and our discussion in the first paragraph as by assumption both x, y is bigger than M.

     So we have proved that there exists an uniform choice of δ for all x in $[a, \infty)$ which makes the continuity condition hold. Therefore f is uniformly continuous.

   - If $b \in \mathbb{R}$ and $a = -\infty$ the proof is very much similar as in the previous bullet point. We just have to cap off a compact set on the left side rather on the right and everything else follows almost verbatim.

   - If $a = -\infty$ and $b = +\infty$. Let the limits at a and b be L_1 and L_2 respectively.Fix $\varepsilon > 0$ and find M and N satisfying $M > N$ such that for $x > M$ we have $|f(x) - L_2| < \frac{\varepsilon}{2}$ and for $x < N$ we have $|f(x) - L_1| < \frac{\varepsilon}{2}$. Choose $\delta \in (0, 1)$ so that the uniform continuity condition holds in the compact interval $[N - 1, M + 1]$.

Then if any of x, y is located outside of that interval, then the uniform continuity condition holds by the very same argument as in the second bullet point.

4. Consider the following function

$$f(x) = \begin{cases} x & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Clearly f is bounded between 0 and 1 and one can see that the infimum value of $\frac{1}{4}$ and the supremum value of $\frac{3}{4}$ are of course not attained as the inequalities are strict in the definition.

This does not contradict Thm. 8.5 because f is not continuous.

5. Note that the given polynomial is of even degree therefore it has the property that $\lim_{x\to\pm\infty} P(x) = +\infty$. As a consequence we can choose $-M < 0$ such that $P(x) > a_0$ for all $x < -M$ and similarly we can choose $N > 0$ such that $P(x) > a_0$ for all $x > M$. Now consider the compact interval $[-M, N]$ in which P is uniformly continuous (this is because all polynomials are continuous (Ex. 8.1.2) and then by Thm 8.11 it is uniformly continuous on any closed interval). Now by Thm 8.5 P must attain its minimum on the compact set $[-M, N]$. This minimum is necessarily less than or equal to $P(0) = a_0$ as $0 \in [-M, N]$. By construction, outside of $[-M, N]$ we have $P(x) > a_0$. Therefore the minimum inside $[-M, N]$ is the minimum of P over the whole of $\mathbb{R}$ and it is attained.

6. Given $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period c. The periodicity implies that the range of f is the same as the range of f when restricted to the interval $[0, c]$. Since $[0, c]$ is compact and f continuous we see by Thm 8.2 (which states that a continuous function takes compact sets to compact sets) that the range of f is compact too. Also by Thm 8.5 f restricted to $[0, c]$ is bounded and attains it bounds therefore the same holds true over all values of f.

   All that remains to show is that f is uniformly continuous. Clearly f restricted to $[-c, c]$ is uniformly continuous (by Thm 8.11 as $[-c, c]$ is compact). Fix $\varepsilon > 0$. By uniform continuity then there exists $\delta > 0$ (we shall also choose δ such that $\delta < c/2$) such that for all $x, y \in [-c, c]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. We'll show that this in fact holds for all $x, y \in \mathbb{R}$. If $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta < c/2$ (and WLOG assume $x < y$) then there either there exists N such that both $x, y \in [Nc, (N+1)c]$ or $x \in [(N-1)c, Nc]$ and $y \in [Nc, (N+1)c]$.

   In both the cases then $(x - Nc), (y - Nc)$ lie in the interval $[-c, c]$. Now $|(x - Nc) - (y - Nc)| = |x - y| < \delta$ implies $|f(x - Nc) - f(y - Nc)| < \varepsilon$ by uniform continuity on $[-c, c]$ as we have assumed before. However, by periodicity, $f(x - Nc) = f(x)$ and $f(y - Nc) = f(y)$, which proves our assertion.

7. (a) YES. Apply Problem 6, and the fact that $\sin x$ is periodic with period $c = 2\pi$.

   (b) YES. Recall the inequality $\big||x| - |y|\big| \leq |x - y|$ (from PS2.7). This implies that f is Lipschitz and so also uniformly continuous by Lem. 8.13.

(c) YES. Recall that $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$. Therefore invoking Problem 3, we conclude that f is uniformly continuous.

(d) NO. We shall directly negate the uniform continuity condition. Fix $\varepsilon = \frac{1}{2}$. We shall show that for any $\delta > 0$ we can find a pair of x, y such that $|x - y| < \delta$ but $|f(x) - f(y)| > \varepsilon$.

Consider a sequence $x_n = \frac{1}{2n}$ and $y_n = \frac{1}{(2n+1)}$. Clearly $|x_n - y_n| \to 0$. But

$$\begin{aligned} |f(x) - f(y)| &= |\cos(\frac{1}{2n})\cos(2n\pi) - \cos(\frac{1}{(2n+1)})\cos((2n+1)\pi)| \\ &= |\cos(\frac{1}{2n}) + \cos(\frac{1}{(2n+1)})| \xrightarrow[n\to\infty]{} 2 \end{aligned} \tag{1}$$

.

So given any $\delta > 0$ choose N_1 such that for all $n > N_1$ we have $|x_n - y_n| < \delta$ and choose N_2 such that for all $n > N_2$ we have $|f(x_n) - f(y_n)| > \frac{1}{2}$. Now choosing some $N > \max\{N_1, N_2\}$ we can say that $|x_N - y_N| < \delta$ but $|f(x_N) - f(y_N)| > \varepsilon$ as desired.

(e) YES. Again note that $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 1^-} f(x) = \frac{1}{e}$. Therefore Problem 3 tells us that f is uniformly continuous.

8. Consider $\mathbb{Z} \subset \mathbb{R}$. We know $\mathbb{Z}$ is non compact as it is unbounded. Consider any function $f : \mathbb{Z} \to \mathbb{R}$. Fix some $\varepsilon > 0$. Choose $\delta = \frac{1}{2}$. Then $|x - y| < \frac{1}{2}$ implies $x = y$ because $x, y \in \mathbb{Z}$. Then $|f(x)-f(y)| = |f(x)-f(x)| = 0 < \varepsilon$ trivially (this is the same property as mentioned at Def. 7.5 that any function is continuous at any isolated point). This shows that any $f : \mathbb{Z} \to \mathbb{R}$ is continuous and since the choice of $\delta = \frac{1}{2}$ is obviously independent of x any such function is uniformly continuous too.

9. The forward implication that f is continuous implies f is uniformly continuous on every compact subset is just Thm 8.11. So we won't prove it again. Let us prove the opposite implication.

   Assume f is continuous on every compact set. We shall prove that f is continuous on X using the sequential criterion. Consider any $x_n \to x$ in X. Consider the compact set $K = \{x_1, x_2, \cdots, x_n, \cdots\} \cup \{x\}$. K is compact for the same reason as A in PS3.10 is; namely that any open covering of K must contain an open set containing x and hence an open ball around x. This open ball by property of convergent sequence must contain all but finitely many x_n's. Choose open sets, one for each of finitely many elements left out. Then this collection along with the one we have chosen for x gives us a finite sub-cover. By assumption f is (uniformly) continuous on K gives us that $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$, which proves that f is continuous.

10. Fix $\varepsilon > 0$. Since f is uniformly continuous on $(a, b]$ we have $\delta_1 > 0$ such that whenever $|u - v| < \delta_1$ for some $u, v \in (a, b]$, we have $|f(u) - f(v)| < \frac{\varepsilon}{2}$, similarly uniform continuity on $[b, c)$ gives us that we have $\delta_2 > 0$ such that whenever $|u - v| < \delta_2$ for some $u, v \in [b, c)$, we have $|f(u) - f(v)| < \frac{\varepsilon}{2}$.

Choose $\delta = \min\{\delta_1, \delta_2\}$. Consider $x, y \in (a, c)$ satisfying $|x - y| < \delta$. If both x, y lie in $(a, b]$ or $[b, c)$ we are done by the assumption of uniform continuity. If not, assume $x < y$ so that $x \in (a, b]$ and $y \in [b, c)$. Then $\delta > |x - y| = y - x = (y - b) + (b - x)$. Since both the terms on the RHS are positive we can conclude $(y - b) < \delta < \delta_2$ and $(b - x) < \delta < \delta_1$. Therefore,

$$|f(x) - f(y)| < |f(x) - f(b)| + |f(y) - f(b)| < \varepsilon$$

proving uniform continuity.

11. Let $A = \mathbb{N}$ and $B = \{n + \frac{1}{n} \mid n \in \mathbb{N}\}$. Clearly A, B are both closed because their complements are unions of open intervals and hence open. Define

$$f := \begin{cases} 1 & \text{if } x \in A \\ 2 & \text{if } x \in B \end{cases}$$

Clearly restricted to A and B the function is uniformly continuous (by exactly the same argument as in Problem 8). Now we shall show that the whole function is not uniformly continuous. Choose $\varepsilon = \frac{1}{2}$. Given any $\delta > 0$ choose $N \in \mathbb{N}$ so that $N > \frac{1}{\delta}$. Consider the pair $x = N$, $y = N + \frac{1}{N}$. Clearly $x \in A$ and $y \in B$ and they satisfy $|x - y| = \frac{1}{N} < \delta$. However, by definition of f, $|f(x) - f(y)| = |1 - 2| = 1 > \frac{1}{2}$.

So we have exhibited that for $\varepsilon = \frac{1}{2}$, given any $\delta > 0$ we can we can find a pair $x, y \in A \cup B$ such that $|x - y| < \delta$ but $|f(x) - f(y)| > \varepsilon$ thus negating uniform continuity.