Math 425A Problem Sheet 9 (due 9am on Monday, 1st Nov)

Essential problems

1. (3 pts) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (i.e. the claim of Cor. 9.4 is valid; namely that that for every $c \in [f(a), f(b)]$ there exists $x \in [a,b]$ such that $f(x) = c$) and $f^{-1}(\{q\})$ is closed for every $q \in \mathbb{Q}$ then f is continuous. *(Comment: Note that the second condition alone is not sufficient for continuity by PS7.1(c).)*

2. (2 pt) Consider

$$f(x) := \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(0) = 1$, but that it is not increasing on any neighbourhood of 0. Why does it not contradict Lem. 9.13?

3. (3 pt) Suppose that $f \in C([0,1]) \cap D((0,1))$ is such that $f(0) = f(1) = 0$ and that $f(x_0) = 1$ for some $x_0 \in (0,1)$. Prove that $|f'(c)| > 2$ for some $c \in (0,1)$.

4. (2 pt) Use the Generalized Mean Value Theorem to show that $1 - x^2/2 < \cos x$ for $x \neq 0$. Deduce that $x - x^3/6 < \sin x$ for $x > 0$.

Additional problems

5. (1 pt)

(a) Are the sets $A := (0,1)$, $B := \mathbb{Z}$ separated?
(b) Are the sets $A := (-\infty, 0)$, $B := \{x \in \mathbb{R} \setminus \mathbb{Q} \colon x > 0\}$ separated?
(c) Is $\mathbb{Q}^2$ a connected set? Is it path-connected?

6. (1 pt)

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period a. Show that there exists $y \in \mathbb{R}$ such that

$$f\left(y + \frac{a}{2}\right) = f(y).$$

Deduce that there are in fact infinitely many such y's.
(b) Show that the equation $\sin(\cos x) = x$ has exactly one solution in $[0, \pi/2]$.

7. (1 pt) Show that if $f\colon [a,b] \to \mathbb{R}$ is nondecreasing and has the intermediate value property, then f is continuous.

8. (1 pt) Suppose that $f\colon (a,b) \to \mathbb{R}$ is differentiable at $x \in (a,b)$. Find

$$\lim_{t \to x} \frac{tf(x) - xf(t)}{t - x}.$$

9. (1 pt) Show that $f(x) := [x]\sin^2(\pi x)$ is differentiable on $\mathbb{R}$. Why can't one use the product rule (Lem. 9.10.2) to calculate f'? Deduce that the function $g(x) := [x]\sin(2\pi x)$ has the intermediate value property.

10. (1 pt) Show that if $f\colon (a,b) \to \mathbb{R}$ attains a local maximum at (a,b), and the one-sided derivatives $f'_-(x) := \lim_{y\to x^-}(f(y)-f(x))/(y-x)$ and $f'_+(x) := \lim_{y\to x^+}(f(y)-f(x))/(y-x)$ exist then $f'_-(x) \geq 0$ and $f'_+(x) \leq 0$. *(Comment: This is a generalization of Thm. 9.14 to the case when $f'(x)$ does not exist.)*

11. (1 pt) Let α be a real number. Show that if $f \in C([a,b]) \cap D((a,b))$ and $f(a) = f(b) = 0$ then there exists $x \in (a,b)$ such that $\alpha f(x) + f'(x) = 0$. *(Comment: This reduces to Rolle's theorem (Thm. 9.15) when $\alpha = 0$; also hint: consider $g(x) := \mathrm{e}^{ax} f(x)$.)*

WOJCIECH OŻAŃSKI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, USA.
Email address: ozanski@usc.edu