MATH 425a PS 9

1. 1) f has IVT 2) $\forall q \in \mathbb{Q}$ $f^{-1}(\{q\})$ is closed

Suppose for contradiction, we have f discontinuous at x.

~~$\exists \varepsilon>0\ \forall \varepsilon$~~ $\hookrightarrow \exists \varepsilon>0$ ~~&~~ $\forall \underset{>0}{\delta}$ s.t. $|x-y|<\delta$ but $|f(x)-f(y)|\geq \varepsilon$.

let $\delta_n=\frac{1}{n}$. for each n, pick $\{y_n\}$ with $|x-y_n|<\delta_n=\frac{1}{n}$. $\Rightarrow \forall n, |f(x)-f(y_n)|\geq \varepsilon$.

We now have: ~~either f(x)~~ WLOG $\exists$ a subsequence $\{y_{n_k}\}$ s.t.

$y_{n_k}>x$, $y_{n_k}-x<\frac{1}{n_k}$, $|f(y_{n_k})-f(x)|\geq \varepsilon$.

WLOG $\exists$ a further subsequence $\{y_{n_{k_j}}\}$ s.t.

$y_{n_{k_j}}>x$, $y_{n_{k_j}}-x<\frac{1}{n_{k_j}}$, $|f(y_{n_{k_j}})-f(x)|\geq\varepsilon \Rightarrow f(y_{n_{k_j}})\geq f(x)+\varepsilon$.

Now pick $q\in(f(x), f(x)+\varepsilon)$. By IVT, $\because f(y_{n_{k_j}})\geq f(x)+\varepsilon \Rightarrow \exists c\in[x, y_{n_{k_j}}]$ s.t. $f(c)=q$.

As $j\to\infty$, $x\leq c\leq y_{n_{k_j}}$, $c\to x$ (because $y_{n_{k_j}}\to x$ as ~~$y\to x$~~ $y_n\to x$. This follows from ~~on this~~ $|x-y_n|<\delta_n$).

$\because f^{-1}(\{q\})$ closed, $\therefore x$ in $f^{-1}(\{q\})$ (x is a limit point).

This ~~$q=f(x)$~~ $f(x)=q>f(x)$, Σ. f must be continuous.

2. ~~$f'(x)=1+1=1-2\cos\frac{1}{x}+4x\sin\frac{1}{x}$~~. $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$

~~f' $\lim_{x\to0^+}f(x)=\lim_{x\to0}1=$~~ $=\lim_{x\to 0}\frac{x+2x^2\sin(\frac{1}{x})}{x}=\lim_{x\to0}(1+2x\sin\frac{1}{x})=1+0=1$.

Not increasing on any neighborhood of 0:

We know $f'(x)=1-2\cos\frac{1}{x}+4x\sin\frac{1}{x}$. we define $\{x_n\}$: $\{\frac{1}{2\pi}, \frac{1}{4\pi}, \frac{1}{6\pi}, \dots\}$.

we see $x_n\to0$, and $\forall\varepsilon>0$ $\exists n>N$ s.t. ~~x_n~~ $0<x_n<\varepsilon$ and $f'(x_n)<0 \Rightarrow$ decreasing.

Not contradict with q:13: there is an interval ~~$(x-\delta$~~ $(-\delta,\delta)$ s.t. $f(y)<0<f(z)$.

In this case, this means infinite roots ~~on~~ of this function on ~~$y=0$~~ ~~y~~ x axis.

3. By MVT, $\exists$ ~~$y\in[0,x_0]$~~ $y\in(0,x_0)$ s.t. $f'(y)=\frac{1}{x_0}$. $\exists z\in(x_0,1)$ s.t. $f'(z)=\frac{-1}{1-x_0}=\frac{1}{x_0-1}$.

By Darboux property $\exists c\in(y,z)$ s.t. $f'(c)=\lambda$ for $\lambda\in(\frac{1}{x_0},\frac{1}{x_0-1})$.

~~We~~ $\because 0<x_0<1$, we have either $\frac{1}{x_0}>2$ or $\frac{1}{x_0-1}<-2$. (if $x_0=\frac{1}{2}$, not differentiable)

$\hookrightarrow \exists c\in(0,1)$ s.t. $|f'(c)|>2$.

4. let $f(x)=\frac{x^2}{2}+\cos x$. $f(0)=1$. $\exists c\in(0,x)$ s.t. $f'(c)=\frac{f(x)-f(0)}{x}$

$\hookrightarrow f(x)-f(0)=x(c-\sin c)$. $\begin{cases} x>0 \to f(x)-f(0)>0 \text{ as } c-\sin c>0 \\ x<0 \to f(x)-f(0)>0 \text{ as } c-\sin c<0\end{cases} \Rightarrow f(x)-f(0)>0$.

$\therefore \frac{x^2}{2}+\cos x>1 \Rightarrow 1-\frac{x^2}{2}<\cos x$.

let $f(x)=$ ~~$\sin x-x$. $f(0)=0$. $\sin x - x$~~, $x-\sin x-\frac{1}{6}x^3$.

$f'(x)=1-\cos x-\frac{1}{2}x^2$, $f''(x)=\sin x-x$. we know if $x>0$, $f''(x)<0$.

$\hookrightarrow f'(x)\leq f'(0)=0$ ~~if~~ $\Rightarrow f(x)\leq f(0)=0 \Rightarrow x-\sin x-\frac{1}{6}x^3<0$, $x-\frac{1}{6}x^3<\sin x$.

5- a) No.

b) Yes.

c) No, No.

7- let f not continuous. $\Rightarrow \exists \delta>0\ \forall \varepsilon>0$ s.t. $|f(x)-f(y)| \geq \varepsilon$ even if $|x-y|<\delta$.

WLOG pick $\{y_n\} \downarrow x$ ~~as~~ for a fixed x (as f non-decreasing). pick $\alpha \in (f(x), f(x)+\varepsilon)$.

By IVT ~~$\exists c \in (x$~~, $\exists c$ s.t. $f(c)=\alpha$, $c>x$.

However, $\exists n>N$ s.t. $x<y_n<c$, as $y_n \to x$.

↳ from discontinuity, ~~$f(y_n) \geq f(x), f(x$~~ $f(y_n) \geq f(x)+\varepsilon > f(c)$.

$c>y_n$. $f(c)<f(y_n)$. ↯.

Thus f must be continuous.

8- $\lim_{t\to x} \frac{t f(x) - x f(t)}{t-x} = \lim_{t \to x} \frac{(-x)(f(t)-f(x)) + (t-x) f(x)}{t-x} = (-x) f'(x) + f(x)$.