Problem 1

Show that if $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property, and $f^{-1}(\{q\})$ is closed for every $q \in \mathbb{Q}$, then f is continuous.

Assume for contradiction that f is not continuous at x_0. By (Heine) there exists a sequence $(x_n) \subset \mathbb{R}$ and $\varepsilon > 0$ such that $x_n \to x_0$ as $n \to \infty$, but $|f(x_n) - f(x_0)| > \varepsilon$. We can assume $f(x_n) < f(x_0) - \varepsilon$ for all n (as there are either infinitely many elements of (x_n) that are less than $f(x_0) - \varepsilon$ or there are infinitely many elements that are larger than $f(x_0) + \varepsilon$; WLOG we assume the former, and we extract an appropriate subsequence of such elements).

By PS2.2(a) there exists $q \in \mathbb{Q}$ such that $q \in (f(x_0) - \varepsilon, f(x_0))$. Applying the assumption about the intermediate value property to the interval with end-points x_0 and x_n we can find z_n located between x_0 and x_n such that $f(z_n) = q$, that is $z_n \in f^{-1}(\{q\})$. Since $|x_n - x_0| \to 0$ as $n \to \infty$ the squeeze theorem (Cor. 5.9) we must also have $|z_n - x_0| \to 0$. That is $(z_n) \subset f^{-1}(\{q\})$ converges to x_0, and so the assumption gives that $x_0 \in f^{-1}(\{q\})$. In other words $f(x_0) = q$, which contradicts our choice of q (that $q < f(x_0)$). Thus f is continuous.

Problem 2

Consider

$$f(x) := \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(0) = 1$, but that it is not increasing on any neighborhood of 0. Why does it not contradict Lem. 9.13?

By definition,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x + 2x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} \left(1 + 2x \sin \frac{1}{x}\right) = 1.$$

Observe that for any $x \neq 0$, we can use the product rule and the chain rule to say that

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x},$$

so for every $x_n := (2\pi n)^{-1}$ for $n \in \mathbb{Z}$ we have that $f'(x_n) = -1$. Thus by Lem. 9.13 there exists $0 < y_n < x_n$ such that $f(y_n) > f(x_n)$.

Hence f cannot be increasing in $(-\epsilon, \epsilon)$ for any $\epsilon > 0$ (if it were, then take n large enough to make sure that $x_n, y_n \in (-\epsilon, \epsilon)$ to obtain a contradiction).

It does not contradict Lem. 9.13, as the lemma only allows to compare the values of f at sufficiently close neighbours of x (where $f'(x) > 0$) with $f(x)$. It does not claim existence of an open interval where f is increasing (i.e. each two values can be compared).

Problem 3

Suppose that $f \in C([0,1]) \cap D((0,1))$ is such that $f(0) = f(1) = 0$ and that $f(x_0) = 1$ for some $x_0 \in (0,1)$. Prove that $|f'(c)| > 2$ for some $c \in (0,1)$.

There are 3 possibilities for the location of x_0: (i) $x_0 \in (0,1/2)$; (ii) $x_0 \in (1/2,1)$; and (iii) $x_0 = 1/2$. In case (i) or (ii), we have either $|f(x_0)/x_0| > 2$ or $|f(x_0)/(1-x_0)| > 2$, so by the Mean Value Theorem, we have $c \in (0,x_0)$ in the former case, and $c \in (x_0,1)$ in the later case, such that $|f'(c)| > 2$.

Suppose now that $x_0 = 1/2$, and assume for contradiction that $|f'(c)| \leq 2$ for every $c \in (0,1)$. Consider $g(x) = 2x$ on $[0,1/2]$. If $f \neq g$, then there exists a point $x_1 \in (0,1/2)$ such that either $f(x_1) > g(x_1)$ or $f(x_1) < g(x_1)$.

If $f(x_1) > g(x_1)$, then $f(x_1) > 2x_1$ which implies $f(x_1)/x_1 > 2$, hence by the Mean Value Theorem, there must be some point $c \in (0,x_1)$ such that $f'(c) > 2$.

If $f(x_1) < g(x_1)$, then $f(x_1) < 2x_1$, hence

$$2 = \frac{1-2x_1}{1/2-x_1} < \frac{f(1/2)-f(x_1)}{1/2-x_1},$$

hence by the Mean Value Theorem, there exists some $c \in (x_1,1/2)$ such that $f'(c) > 2$. Thus we must have that $f(x) = 2x$ on $[0,1/2]$.

In the same we show that $f(x) = 2-2x$ on $[1/2,1]$. However, such function f is not differentiable at $1/2$, which contradicts the assumption. Thus there exists c such that $|f'(c)| > 2$.

Problem 4

Use the Generalized Mean Value Theorem to show that $1 - x^2/2 < \cos x$ for $x \neq 0$. Deduce that $x - x^3/6 < \sin x$ for $x > 0$.

Note that since $\cos x$ and $1-x^2/2$ are both even, it suffices to show the result on $(0,\infty)$. Furthermore, since $1-x^2/2 < -1$ for $x > 2$ and since $\cos x \geq -1$, we only need to show that $\cos x > 1-x^2/2$ on $(0,1]$. Applying the Generalized MVT (Thm. 9.16) with $f(x) := 1-\cos x$, $g(x) := x^2/2$ we see that there exists $a \in (0,1)$ such that

$$\frac{1-\cos x}{x^2/2} = \frac{\sin(a)}{a}$$

Applying GMVT again with $f(x) := \sin(x)$, $g(x) := x$ we see that there exists $b \in (0,a)$ such that

$$\frac{\sin(a)}{a} = \cos(b),$$

which is less than 1, as $b \in (0,1)$.

As for the sin we apply the GMVT with $f(x) := x - \sin x$ and $g(x) := x^3/6$ to see that there exists $a \in (0, x)$ such that

$$\frac{x - \sin x}{x^3/6} = \frac{1 - \cos a}{a^2/2},$$

which is less than 1 by the claim above.

Problem 5

(a) Are the sets $A := (0, 1)$, $B := \mathbb{Z}$ separated?

(b) Are the sets $A := (-\infty, 0)$, $B := \{x \in \mathbb{R} \setminus \mathbb{Q} : x > 0\}$ separted?

(c) Is $\mathbb{Q}^2$ a connected set? Is it path-connected?

(a) Observe that $\overline{A} = [0, 1]$, so $\overline{A} \cap \mathbb{Z} = \{0, 1\}$. Thus they are not separated.

(b) Observe that B is the set of all positive irrational numbers, hence $\overline{B}$ is the set of all non-negative real numbers, $[0, \infty)$. Since $\overline{A} = (-\infty, 0]$, we have $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, so A and B are separated.

(c) Note that

$$\mathbb{Q}^2 = A \cup B,$$

where

$$A := \{(x, y) \in \mathbb{Q}^2 : x < \sqrt{2}\}, \quad B := \{(x, y) \in \mathbb{Q}^2 : x > \sqrt{2}\}.$$

These are separated sets (as $\overline{A} = \{(x, y) \in \mathbb{R}^2 : x \leq \sqrt{2}\}$ is disjoint with B and analogously $A \cap \overline{B} = \emptyset$), so $\mathbb{Q}^2$ is not connected. It is not path connected due to Lem. 9.1.

Problem 6

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period a. Show that there exists $y \in \mathbb{R}$ such that

$$f\left(y + \frac{a}{2}\right) = f(y).$$

Deduce that there are in fact infinitely many such y's.

(b) Show that the equation $\sin(\cos x) = x$ has exactly one solution in $[0, \pi/2]$.

(a) Let $g(y) := f(y + a/2) - f(y)$. If $g(0) = 0$ then the claim follows by taking $y = 0$. If $g(0) \neq 0$, say $g(0) < 0$ then $g(a/2) = f(a) - f(a/2) = f(0) - f(a/2) = -g(0) > 0$, where we used periodicity of f in the second equality. By the Intermediate Value Theorem (Cor. 9.4) we obtain $y \in (0, a/2)$ such that $g(y) = 0$, which proves the claim. Given a single y that satisfies the claim, by periodicity $y + ka$ also satisfies the claim for each $k \in \mathbb{Z}$, which proves the second claim.

(b) By Ex. 9.5 there exists at least one solution in $[0, \pi/2]$. Thus we need to show that there is only one. For this note that

$$f(x) := \sin(\cos x) - x$$

is such that $f(0) = \sin(1) > 0$ and $f(\pi/2) = -\pi/2 < 0$, so neither 0 nor $\pi/2$ are solutions. Furthermore f is strictly decreasing on $(0, \pi)$ as $f'(x) = -\cos(\cos x)\sin x - 1 < 0$ for $x \in (0, \pi)$ (and using Cor. 9.18), and so it is injective (recall the problem from the discussion session). Thus there cannot be another solution in $(0, \pi/2)$.

Alternatively, let $s := \inf\{x \in [0, \pi/2] : f(x) = 0\}$. The set $\{x : f(x) = 0\}$ is nonempty and closed (by PS7.10), hence $s \in \{x \in [0, \pi/2] : f(x) = 0\}$ (by Ex. 3.11). If t is any other point in $\{x \in [0, \pi/2] : f(x) = 0\}$, then $t > s$, but then $t = \sin(\cos t) \le \sin(\cos s) = s$, which is a contradiction.

Problem 7

Show that if $f : [a, b] \to \mathbb{R}$ is nondecreasing and has the intermediate value property, then f is continuous.

By #1, it suffices to show that $f^{-1}(\{q\})$ is closed for every $q \in \mathbb{Q}$. (In fact, we will show that for every $r \in \mathbb{R}$, $f^{-1}(\{r\})$ is closed.)

Let $r \in [f(a), f(b)]$ be given. By the intermediate value property, $f^{-1}(\{r\}) \neq \emptyset$. Put $s := \inf f^{-1}(\{r\})$ and $t := \sup f^{-1}(\{r\})$. We will show that $[s, t] = f^{-1}(\{r\})$, which is closed. To show that, we need to show that $f(s) = f(t) = r$, and for every u such that $s \le u \le t$, $f(u) = r$. Note that the latter fact follows from the former since f is nondecreasing (therefore $r = f(s) \le f(u) \le f(t) = r$).

We will only show $f(s) = r$, since the proof of $f(t) = r$ is analogous. Assume not. Then since f is nondecreasing, $f(s) < r$, and so by the intermediate value property there exists $c > s$ such that $f(c) \in (f(s), r)$. However, by definition of inf there exists $s' \in f^{-1}(\{r\})$ such that $s' \in (s, c)$. Thus $f(s') = r > f(c)$, which contradicts the assumption that f is nondecreasing.

Problem 8

Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$. Find

$$\lim_{t \to x} \frac{tf(x) - xf(t)}{t - x}.$$

By the definition of derivative,

$$\begin{aligned}\lim_{t\to x}\frac{tf(x)-xf(t)}{t-x} &= \lim_{t\to x}\frac{tf(x)-xf(x)+xf(x)-xf(t)}{t-x}\\ &= f(x)-\lim_{t\to x}\frac{x(f(t)-f(x))}{t-x}\\ &= f(x)-x\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\ &= f(x)-xf'(x).\end{aligned}$$

Note that L'Hôpital is **not** useable here because we don't know if $f'(x)$ is continuous, hence we can't conclude that $\lim_{t\to x} f'(t) = f'(x)$.

Problem 9

Show that $f(x) := [x]\sin^2(\pi x)$ is differentiable on $\mathbb{R}$. Why can't one use the product rule to calculate f'? Deduce that the function $g(x) := [x]\sin(2\pi x)$ has the intermediate value property.

Because $[x]$ is locally constant on every open interval of the form $(n, n+1)$ with $n \in \mathbb{Z}$, it suffices to show that f is differentiable at every $x \in \mathbb{Z}$. By definition,

$$f'(n) = \lim_{x\to n}\frac{[x]\sin^2(\pi x)-n\sin^2(n\pi)}{x-n} = \lim_{x\to n}\frac{[x]\sin^2(\pi x)}{x-n}.$$

We have

$$\lim_{x\to n^+}\frac{[x]\sin^2(\pi x)}{x-n} = n\lim_{x\to n^+}\frac{\sin^2(\pi x)}{x-n} \overset{H}{=} n\lim_{x\to n^+}\frac{2\sin(\pi x)\cos(\pi x)}{1} = 0$$

and similarly

$$\lim_{x\to n^-}\frac{[x]\sin^2(\pi x)}{x-n} = (n-1)\lim_{x\to n^-}\frac{\sin^2(\pi x)}{x-n} \overset{H}{=} (n-1)\lim_{x\to n^-}\frac{2\sin(\pi x)\cos(\pi x)}{1} = 0.$$

Thus $f'(n) = 0$ for all $n \in \mathbb{Z}$, and so

$$f'(x) = [x]2\sin(\pi x)\cos(\pi x) = [x]\sin(2\pi x).$$

We couldn't have used the product rule as $[x]$ is not a differentiable function. Also $f'(x)$ has the intermediate property by the Darboux Theorem (Thm. 9.19).

Problem 10

Show that if $f : (a,b) \to \mathbb{R}$ attains a maximum at (a,b) and the one-sided derivatives $f'_-(x) := \lim_{y\to x^-}(f(y)-f(x))/(y-x)$ and $f'_+(x) := (f(y)-f(x))/(y-x)$ exist, then $f'_-(x) \geq 0$, and $f'_+(x) \leq 0$.

For every $y < x$, we have $f(y) - f(x) \leq 0$, and $y - x \leq 0$, therefore $(f(y) - f(x))/(y - x) \geq 0$. A similar argument can be used to prove that $(f(y) - f(x))/(y - x) \leq 0$ when $y > x$.

Problem 11

Let α be a real number. Show that if $f \in C([a,b]) \cap D((a,b))$ and $f(a) = f(b) = 0$, then there exists $x \in (a,b)$ such that $\alpha f(x) + f'(x) = 0$.

Let α be given and consider $g(x) = e^{\alpha x} f(x)$. Then $g(a) = g(b) = 0$, and $g'(x) = \alpha e^{\alpha x} f(x) + e^{\alpha x} f'(x)$. By Rolle's theorem, there exists some $x \in (a,b)$ such that $g'(x) = \alpha e^{\alpha x} f(x) + e^{\alpha x} f'(x) = 0$. Dividing both sides by $e^{\alpha x}$, we obtain $\alpha f(x) + f'(x) = 0$.