Rudin RCA Chapter 1 Exercises*

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Problem 1

Does there exist an infinite σ-algebra which has only countably many members?

Solution. No. Proof in a nutshell: $\mathcal{P}(\mathbb{N})$ is uncountable [$\mathcal{P}(\mathbb{N})$ denoting the power set of $\mathbb{N}$].

Proof. Let $\mathfrak{M}$ be any infinite σ-algebra on X. Since $\mathfrak{M}$ is infinite, X clearly cannot be finite as the power set of a finite set is still finite. Also, by definition, $\varnothing, X \in \mathfrak{M}$. Now we define $S_1 := \varnothing, S_2 := X$, and (assuming AC)

$$S_{n+1} := \text{ any set in } \mathfrak{M} - \{S_1, \ldots, S_n\}.$$

This is well-defined because each S_i is finite while $\mathfrak{M}$ is infinite. It is also clear that S_{n+1} is disjoint from $S_1, \ldots, S_n$, so in particular the collection of all S_i's are pairwise disjoint.

Now we consider a map $f : \mathcal{P}(\mathbb{N}) \to \mathfrak{M}$ by mapping $T \subset \mathbb{N}$ to $\bigcup_{i \in T} S_i$, i.e., the union of all S_i's whose indices are in T. Since S_i's are disjoint, this map is injective, so it follows that $|\mathfrak{M}| \geqslant |\mathcal{P}(\mathbb{N})|$, and thus no infinite σ-algebra can contain only countably many members. □

Problem 2

Prove the analogue of Theorem 1.8 for n functions: if $u_1, \ldots, u_n$ are real measurable functions on a measurable space X, if Φ is a continuous mapping from $\mathbb{R}^n$ to a topological space Y, and if we define

$$h(x) := \Phi(u_1(x), \ldots, u_n(x)),$$

then $h : X \to Y$ is measurable.

Proof. The only thing we need to modify is the following identity:

$$f^{-1}(R) = \bigcup_{i=1}^{n} u_i^{-1}(I_i)$$

where $R = \prod_{i=1}^{n} I_i \in \mathbb{R}^n$ is any open box. Everything else follows identically. □

*We will adopt Rudin's notations, e.g., $\mathfrak{M}$ for σ-algebra and "$-$" for difference of sets. Exceptions do apply (e.g. $\mathbb{R}$).

Problem 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geqslant r\}$ is measurable for every rational r, then f is measurable.

Proof. Let $V = (a,b) \in \mathbb{R}$ be any open set; we aim to prove that $f^{-1}(V)$ is measurable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist *decreasing rational* sequences $\{a_n\}_{n\geqslant 1}, \{b_m\}_{m\geqslant 1}$ such that

$$\lim_{n,m\to\infty}(a_n, b_m) \to (a,b).$$

We also define $A_n := \{x : f(x) \geqslant a_n\}$ and B_m likewise. By assumption all of these sets are measurable, by Definition 1.3(c) $\bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{m=1}^{\infty} B_m$ are measurable. Using Definition 1.3(b), (c), and Comment 1.6(c),

$$\bigcup_{n=1}^{\infty} A_n \cap \Big(X - \bigcup_{m=1}^{\infty} B_m\Big)$$

is also measurable, and it remains to notice that this is simply $f^{-1}(a,b) = f^{-1}(V)$:

$$\begin{aligned} f^{-1}(a,b) &= f^{-1}((-\infty,b)\cap(a,\infty)) \\ &= f^{-1}((a,\infty)\cap[\mathbb{R}-(b,\infty)]) \\ &= f^{-1}\Big(\bigcup_{n=1}^{\infty}(a_n,\infty)\cap\Big[\mathbb{R}-\bigcup_{m=1}^{\infty}(b_m,\infty)\Big]\Big) \\ &= \bigcup_{n=1}^{\infty} A_n \cap \Big(X - \bigcup_{m=1}^{\infty} B_m\Big). \end{aligned}$$

□

Problem 4

Let $\{a_n\}, \{b_n\}$ be sequences in $[-\infty, \infty]$. Prove that

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$ and

(b) $\limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, assuming the sums are not of forms $\infty - \infty$, and

(c) if $a_n \leqslant b_n$ for all n then $\liminf_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} b_n$.

Also show by example that strictly inequality can hold in (b).

Proof. (a) It suffices to show that $\sup a_n = -\inf(-a_n)$ (since a sequence has a unique limit). If $\{a_n\}$ is unbounded from above then $\{-a_n\}$ is unbounded from below and the claim follows trivially.

Now we assume $\{a_n\}$ is bounded from above, so $\inf(-a_n)$ exists. On one hand $a_k \leqslant \sup\{a_n\}$ for all k, so $-a_k \geqslant -\sup\{a_n\}$, implying $-\sup\{a_n\} \leqslant \inf\{-a_n\}$. On the other hand, each $(-a_k) \geqslant \inf\{-a_n\}$ so $a_k \leqslant -\inf\{-a_n\}$ for all k, implying that $\sup\{a_n\} \leqslant -\inf\{a_n\}$. Combining these two inequalities we obtain the desired equality and hence (a).

(b) The statement becomes immediate if we use another dummy variable for $\{b_n\}$:

$$\begin{aligned}\limsup_{n\to\infty}(a_n+b_n) &= \lim_{n\to\infty}\sup_{k\geqslant n}(a_k+b_k) \leqslant \lim_{n\to\infty}\sup_{k,\ell\geqslant n}(a_k+b_\ell)\\ &= \lim_{n\to\infty}\left[\sup_{k\geqslant n} a_n + \sup_{\ell\geqslant n} b_\ell\right] = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

For a strict inequality, consider $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then (a_n+b_n) is a sequence of zeros whereas $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$.

(c) If $\{b_n\}$ dominates $\{a_n\}$ then the sequence $\left\{\sup_{k\geqslant n} b_k\right\}$ dominates $\left\{\sup_{k\geqslant n} a_n\right\}$, and taking limit gives the desired inequality. □

Problem 5

(a) Suppose $f, g : X \to [-\infty,\infty]$ are measurable. Prove the following sets are measurable:

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Proof. (a) By Theorem 1.9(c), $f-g$ is measurable. Hence $(f-g)^{-1}\{[-\infty,0)\}$ and $(f-g)^{-1}(0)$ are measurable (recall that $[-\infty,a)$ is declared open). *Alternatively, for the singleton,* $\{0\}$ *is the complement of* $[-\infty,0)\cup(0,\infty]$ *in* $[-\infty,\infty]$*, so the preimage is also measurable.*

(b) For a fixed $x\in X$, $\lim_{n\to\infty} f_n(x)$ exists if and only if its limsup and liminf agree. We proceed and define $\overline{f}$ to be the pointwise limsup of $\{f_n\}$ and $\underline{f}$ to be the pointwise liminf of $\{f_n\}$. Theorem 1.14 states that $\overline{f}$ and $\underline{f}$ are measurable, so by (a) we know the set $\{x : \overline{f}(x) = \underline{f}(x)\}$ is measurable. This proves the claim. □

Problem 6

Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E\subset X$ such that either E or E^c is at most countable, and define $\mu(E)=0$ in the first case, $\mu(E)=1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measure functions and their integrals.

Proof. We first show that $\mathfrak{M}$ is a σ-algebra.

(a) X is uncountable whereas $X^c = \varnothing$ is countable. ✓

(b) If $A\in\mathfrak{M}$ then either A or A^c is (at most) countable. Therefore either A^c or $(A^c)^c$ is countable. ✓

(c) Let $\{A_n\}$ be such that either A_n or A_n^c is countable for each n. If each A_i is countable then $A = \bigcup_{n=1}^{\infty} A_n$ is the countable union of countable sets and is therefore countable. On the other hand, if one of A_i's is uncountable then the corresponding A_i^c is countable, and by De Morgan's law,

$$A^c = \bigcap_{n=1}^{\infty} A_n^c \subset A_i^c$$

which must also be countable. Either way, $A \in \mathfrak{M}$. ✓

To verify μ is a measure, we need to check if it is countably additive. Let $\{A_n\}$ be disjoint members of $\mathfrak{M}$ and again define A to be their (disjoint) union. If A is countable then so is each A_i, so

$$\mu(A) = \mu(A_i) = 0 \implies \mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

On the other hand, if A is uncountable then $\mu(A) = 1$ and A^c is countable. We claim that *exactly* one A_i is uncountable. If A_i, A_j are both uncountable with $i \neq j$, then A_i^c, A_j^c are both countable; furthermore

$$(A_i^c \cup A_j^c) = (A_i \cap A_j)^c = \varnothing^c = X,$$

so X is countable, which is absurd. Therefore all but one A_i's is countable, and $\mu(A) = 1 = \sum_{n=1}^{\infty} \mu(A_n)$.

A measurable function from $X \to \mathbb{R}$ is one such that, for all $r \in \mathbb{R}$, either $f^{-1}(r)$ or $(X - f^{-1}(r))$ is countable. If we write $E \subset \mathbb{R}$ to be the collection of real numbers of the second type, then

$$\int_X f \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu = \int_{x \in E} \mathrm{d}x.$$

□

Problem 7

Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$; $f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$, $f_n(x) \to f(x)$ as $n \to \infty$ pointwise, and $f_1 \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

and show that this conclusion does *not* follow without the condition "$f_1 \in L^1(\mu)$".

Proof. With the L^1 condition, this directly follows the Lebesgue DCT (Theorem 1.34) with f_1 as global bound. As a "counterexample", notice that if $\mu(\{x \in X : f_1(x) = \infty\}) > 0$ then f_1 is not Lebesgue. Consider

$$f_n : \mathbb{R} \to \mathbb{R} \quad \text{defined by} \quad x \mapsto \begin{cases} \infty & 0 \leqslant x \leqslant 1/n \\ 0 & \text{otherwise.} \end{cases}$$

It follows that no function in this sequence is L^1 (and thus has infinite integral) whereas the entire sequence converges pointwise to the zero function, of which the integral is indeed 0. □

Problem 8

Put $f_n = \chi_E$ if n is odd and $f_n = 1 - \chi_E$ if n even. What is the relevance of this example to Fatou's lemma?

Solution. This example shows that strict inequality can be obtained in Fatou's lemma. The liminf of $\{f_n\}$ is the zero function, but the liminf of the integrals is $\min(\mu(E), \mu(X - E))$ which can be nonzero if chosen properly.

Problem 9

Suppose μ is a positive measure on X, $f : X \to [0, \infty]$ is measurable, $\int_X f \, \mathrm{d}\mu = c$ where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty}\int_X n\log[1+(f/n)^\alpha]\,\mathrm{d}\mu = \begin{cases}\infty & 0<\alpha<1\\ c & \alpha=1\\ 0 & 1<\alpha<\infty.\end{cases}$$

Hint: if $\alpha \geqslant 1$, the integrands are dominated by αf. If $\alpha < 1$, use Fatou's lemma.

Proof. By L'Hôpital's rule,

$$\lim_{n\to\infty} n\log[1+(f/n)^\alpha] = \lim_{n\to\infty}\frac{\log(1+f^\alpha/n^\alpha)}{1/n} = \lim_{n\to\infty}\frac{\alpha n^{1-\alpha}f^\alpha}{1+f^\alpha/n^\alpha}. \qquad (\Delta)$$

If $\alpha = 1$ it is clear that the limit tends to f, so the integral tends to c. If $\alpha < 1$ then the term n^{1-a} (along with $1/n^\alpha$ in the denominator) dominates the integral, so it tends to ∞ as $n \to \infty$. Therefore liminf of the integrand also tends to ∞, and by Fatou's lemma

$$\begin{aligned}\int_X \lim_{n\to\infty} \dots\,\mathrm{d}\mu &= \int_X \liminf_{n\to\infty} \dots\,\mathrm{d}\mu\\ &\leqslant \liminf_{n\to\infty}\int_X \dots\,\mathrm{d}\mu \leqslant \lim_{n\to\infty}\int_X \dots\,\mathrm{d}\mu,\end{aligned}$$

so the last expression must also be ∞. If $\alpha > 1$, by (Δ) we can see that the limit of the integrand tends to 0. On the other hand, using the hint, we can invoke the Lebesgue DCT and get

$$\lim_{n\to\infty}\int_X n\log[1+(f/n)^\alpha]\,\mathrm{d}\mu = \int_X \lim_{n\to\infty} n\log[1+(f/n)^\alpha]\,\mathrm{d}\mu.$$

This concludes the third case and we are done. □

Problem 10

Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu$$

and show that "$\mu(X) < \infty$" cannot be omitted.

Proof. Since $\{f_n\}$ is bounded and uniformly convergent, it follows that $\{\|f_n\|\}$ forms a bounded sequence, and in particular $\|f\|$ is bounded. Because $\mu(X) < \infty$,

$$\int_X f\,\mathrm{d}\mu \leqslant \|f\|\mu(X) < \infty,$$

and invoking the Lebesgue DCT gives

$$\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu.$$

A "counterexample" if we miss the "$\mu(X) < \infty$" assumption: let $f_n \equiv 1/n$ be functions defined on $\mathbb{R}$. Clearly they converge uniformly to the zero function, but the integral of each individual f_n is ∞. □

Problem 11

Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Theorem. Let $\{E_k\}$ be a sequence of measurable sets in X with

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then almost all $x \in X$ lie in finitely many of the sets E_k.

Proof. To show the equation we need to verify both directions. $\leqslant$ is immediate by definition: if $a \in A$ then a lies in all but finitely many [*I believe Rudin's definition of A has a slight logical error when he only says "infinitely many"*] E_k and thus $a \in \bigcup_{k \geqslant n}^{\infty} E_k$ for large enough n's. On the other hand, if $a \notin A$ then it is in at most finitely many E_k's by definition, so it cannot be in the infinite intersection either. Thus the equation holds.

To directly prove Theorem 1.41, we use Theorem 1.19(e). Let $B_n := \bigcup_{k=n}^{\infty} E_k$. Immediately we see that $B_1 \supset B_2 \supset \ldots$, and $\mu(B_1) \leqslant \sum_{k=1}^{\infty} \mu(E_k) < \infty$. Therefore $\mu(B_n) \to \mu(B)$. It remains to notice that $\mu(B_n) \to 0$: indeed,

$$\mu(B_n) \leqslant \sum_{k=n}^{\infty} \mu(E_k) \to 0 \text{ since } \sum_{k=1}^{\infty} \mu(E_k) = 0 \text{ by assumption.}$$ □

Problem 12

Suppose $f \in L^1(\mu)$. Prove that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, \mathrm{d}\mu < \epsilon$ whenever $\mu(E) < \delta$.

Proof. Consider $\{A_n\}$ where $A_n := \{x : |f(x)| \geqslant n\}$. Immediately we see that $A_1 \supset A_2 \supset \ldots$. Since $f \in L^1$ we know $\mu(A_1) < \infty$ and $\mu(A_\infty) := \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$. Therefore the sequence $\{|f_n|\}$ defined by $|f_n| := |f|\chi_{A_n}$ converges to the zero function pointwise. Note that each $|f_n|$ agrees with $|f|$ on some A_n and they are all dominated by $f \in L^1(\mu)$. Therefore, using the Lebesgue DCT,

$$\lim_{n\to\infty} \int_X |f_n| \, \mathrm{d}\mu = \lim_{n\to\infty} \int_{A_n} |f_n| \, \mathrm{d}\mu = \int_X 0 \, \mathrm{d}\mu = 0,$$

and so we can make $\int_E |f| \, \mathrm{d}\mu$ arbitrarily small by choosing an appropriate A_n. □

Problem 13

Show that the proposition 1.24(c) is also true when $c = \infty$: if $f \geqslant 0$ then

$$\int_E cf \, \mathrm{d}\mu = c \int_E f \, \mathrm{d}\mu.$$

Proof. Let $\{c_n\}_{n\geqslant 1}$ be an unbounded, positive seqeuence (so that it tends to ∞). We want to show that

$$\int_E \left(\lim_{n\to\infty} c_n\right) f \, \mathrm{d}\mu = \left(\lim_{n\to\infty} c_n\right) \int_E f \, \mathrm{d}\mu. \tag{Δ}$$

Working on the RHS, we have $\lim_{n\to\infty} c_n \int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E c_n f \, \mathrm{d}\mu$ by the actual 1.24(c) (for finite coefficients).

On one hand, since $\{c_n\}$ is icreasing, $0 \leqslant c_n f \leqslant \lim_{n\to\infty} c_n f$ and so

$$\int_E c_n f \, \mathrm{d}\mu \leqslant \int_E \lim_{n\to\infty} c_n f \, \mathrm{d}x \quad \text{for all } n \implies \lim_{n\to\infty} \int_E c_n f \, \mathrm{d}\mu \leqslant \int_E \lim_{n\to\infty} c_n f \, \mathrm{d}\mu. \tag{$\mathfrak{T}$}$$

On the other hand, by Fatou's lemma,

$$\int_E \lim_{n\to\infty} c_n f \, \mathrm{d}\mu = \int_E \liminf_{n\to\infty} c_n f \, \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_E c_n f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E c_n f \, \mathrm{d}\mu, \tag{$\mathfrak{M}$}$$

where the first and last equalities hold because the infimum of an increasing sequence is equal to the first term. Combining ($\mathfrak{T}$) and ($\mathfrak{M}$) we recover (Δ), completing the proof. □