Rudin RCA Ch1 Theorems

(1.2) (a) A collection τ of subsets of X is said to be a **topology** in X if τ contains $\varnothing, X$, is closed under finite intersection, and is closed under arbitrary union.

(b) If so, (X, τ) is a **topological space** and members of τ are called **open sets** in X.

(c) If X, Y are topological spaces and $f : X \to Y$, then f is **continuous** if $V \subset Y$ open $\Rightarrow f^{-1}(V) \subset X$ open.

(1.3) (a) A collection $\mathfrak{M}$ of subsets of X is said to be a **$\boldsymbol{\sigma}$-algebra** if $\mathfrak{M}$ contains X, is closed under complementation, and is closed under countable union.

(b) If so, $(X, \mathfrak{M})$ is called a **measurable space** and members of $\mathfrak{M}$ are called **measurable sets** in X.

(c) Let X be measurable and Y topological. A function $f : X \to Y$ is said to be **measurable** if $V \subset Y$ open $\Rightarrow f^{-1}(V) \subset X$ measurable.

(1.6) (More on 1.3) $\mathfrak{M}$ contains $\varnothing$ and is closed under *finite* union [1.3(c)] and countable intersection [1.3(b)(c) with De Morgan's]. Also if $A, B \in \mathfrak{M}$ then $A - B = (B^c \cap A) \in \mathfrak{M}$.

(1.7) Let Y, Z be topological spaces and let $g : Y \to Z$ be continuous.

(a) Continuous $\circ$ continuous = continuous: if X is a topological space and if $f : X \to Y$ continuous, then $h := (g \circ f) : X \to Z$ is continuous.

(b) Continuous $\circ$ measurable = measurable: if X is a measurable space and if $f : X \to Y$ measurable, then $h := (g \circ f) : X \to Z$ is measurable.

(1.8) If X is a measurable space, u, v real measurable functions, and $\Phi : \mathbb{R}^2 \to$ some topological space Y, then

$$h : X \to Y \quad \text{defined by} \quad h(x) := \Phi(u(x), v(x))$$

q is measurable. *Proof: check measurability of* $f : x \mapsto (u(x), v(x))$ *using* u^{-1}, v^{-1}. *Then* $h = \Phi \circ f$.

(1.9) Let X be a measurable space.

(a) If u, v are real measurable functions on X then $f : u + iv$ is a complex measurable function: *use* $\Phi(z) = z$.

(b) If $f = u + iv$ is a complex measurable function, then u, v, and $|f|$ are real, measurable functions. *Use* $g(z) := \mathfrak{Re}(z), \mathfrak{Im}(z)$, *and* $|z|$, *respectively.*

(c) If f, g are complex measurable, then so are $f + g$ and fg. *Use* $\Phi(s, t) = s + t$ *and* st, *respectively.*

(d) If E is a measurable set then χ_E is measurable.

(e) If f is complex measurable on X, then there exists a complex measurable α such that $|\alpha| = 1$ and $f = \alpha|f|$. *Use* $\alpha(x) = \varphi(f(x) + \chi_E(x))$ *where* $\varphi(z) = z/|z|$ *and* E *is where* f *vanishes.*

(1.10) Given $\mathscr{F}$ a collection of subsets of X then there exists a smallest σ-algebra $\mathfrak{M}^*$ containing $\mathscr{F}$. *Proof: take all the intersections of* $\mathfrak{M}$ *containing* $\mathfrak{F}$.

(1.11) The **Borel sets** $\mathscr{B}$ of X are members of the smallest σ-algebra $\mathscr{B}$ in X containing every open set in X. In particular, F_σ's and G_δ's, denoting all countable union of closed sets and all countable intersection of open sets, are Borel sets. Every continuous mapping of $(X, \mathscr{B})$ is Borel measurable.

(1.12) Let $\mathfrak{M}$ be a σ-algebra in X, Y a topological space, and $f : X \to Y$.

(a) Ω, the collection of all $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, is a σ-algebra. *Check by definition.*

(b) If f is measurable and E a Borel set in Y then $f^{-1}(E) \in \mathfrak{M}$. Ω *in (a) contains all Borel sets in* Y.

(c) If $Y = [-\infty, \infty]$ and if $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α then f is measurable. *Use limits and intersections.*

(d) If f is measurable, Z a topological space, and $g : Y \to Z$ Borel, then $h := (g \circ f) : X \to Z$ is measurable.

(1.14) If $f_n : X \to [-\infty, \infty]$ is measurable, then

$$g := \sup f_n \quad h := \limsup_{n\to\infty} f_n$$

are measurable. *Take limits on* $g^{-1}((\alpha, \infty])$ *and use 1.12(c). Repeat it with* $\limsup = \inf \sup$.

(1.15) For $f : X \to [-\infty, \infty]$ measurable, define

$$f^+ := \max\{f, 0\} \qquad f^- := -\min\{f, 0\} \quad \text{(pointwise)}$$

to be the **positive** and **negative parts** of f. Then

(a) $|f| = f^+ + f^-$,

(b) $f = f^+ + f^-$, and

(c) if $f = g - h$ with g, h positive, then $f^+ \leqslant g$ and $f^- \leqslant h$.

(1.16) A complex function s on a measurable space X is called a **simple function** if its range consists of finitely many points excluding $\pm\infty$. Let $\alpha_1, ..., \alpha_n$ denote the distinct values of s and let $A_i := \{x : s(x) = \alpha_i\}$. Then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

(1.17) Let $f : X \to [0, \infty]$ be measurable. Then there exist simple functions s_n on X such that

(a) $0 \leqslant s_1 \leqslant ... \leqslant f$, and

(b) $s_n(x) \to f(x)$ pointwise.

(1.18) (a) A **(positive) measure** is a countably additive function μ defined on $\mathfrak{M}$ with range in $[0, \infty]$: if $\{A_i\}$ are disjoint members of $\mathfrak{M}$ then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i).$$

(b) A **measure space** X or $(X, \mathfrak{M}, \mu)$ is a measurable space with a (positive) measure defined on the σ-algebra of its mesurable sets.

(c) A complex measure is defined analogously.

(1.19) Let μ be a (positive) measure on $\mathfrak{M}$. Then

(a) $\mu(\varnothing) = 0$: *take* A *with* $\mu(A) < \infty$*; then* $\mu(A \cup \varnothing \cup \ldots) = \mu(A) + \mu(\varnothing) + \ldots$.

(b) μ is countably additive for finitely many disjoint sets too.

(c) If $A \subset B$ and $A, B \in \mathfrak{M}$ then $\mu(A) \leqslant \mu(B)$: *notice that* $\mu(B) = \mu(A) + \mu(B - A)$.

(d) If $A = \bigcup_{n=1}^{\infty} A_n$ for $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset \ldots$, then $\mu(A_n) \to \mu(A)$. *Define* $B_1 = A_1$ *and* $B_n = A_n - A_{n-1}$ *and use* μ*'s countable additivity.*

(e) If $A = \bigcap_{n=1}^{\infty} A_n$ for $A_n \in \mathfrak{M}$, $A_1 \supset A_2 \supset \ldots$, and $\mu(A_1)$ is finite, then $\mu(A_n) \to \mu(A)$. *Define* $C_n = A_1 - A_n$ *so* $C_1 \subset C_2 \subset \ldots$ *and* $A_1 - A = \bigcup_{n=1}^{\infty} C_n$*. Then use (d).* A non-example when $\mu(A_1) = \infty$: let $\mu(E) = |E|$ and consider $A_n = \{n, n+1, \ldots\}$. Then the nester intersection is $\varnothing$ whereas $\mu(A_n) = \infty$ for each A_n.

(1.23) If $s : X \to [0, \infty)$ is a measurable simple function of form $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, then we define

$$\int_E s \, \mathrm{d}\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).$$

If $f : X \to [0, \infty]$ is measurable (note we can have ∞) and $E \in \mathfrak{M}$, define the **Lebesgue integral**

$$\int_E f \, \mathrm{d}\mu := \sup_{0 \leqslant s \leqslant f} \int_E s \, \mathrm{d}\mu.$$

(1.24) (a) If $0 \leqslant f \leqslant g$ then $\int_E f \, \mathrm{d}\mu \leqslant \int_E g \, \mathrm{d}\mu$,

(b) If $A \subset B$ and $f \geqslant 0$ then $\int_A f \, \mathrm{d}\mu \leqslant \int_B f \, \mathrm{d}\mu$,

(c) If $f \geqslant 0$ and $c \geqslant 0$ a constant (finite or infinite, cf. Ex.1.13) then $\int_E cf \, \mathrm{d}\mu = c \int_E f \, \mathrm{d}\mu$,

(d) If $f(x) = 0$ for all $x \in E$ then $\int_E f \, \mathrm{d}\mu = 0$ even if $\mu(E) = \infty$,

(e) If $\mu(E) = 0$ then $\int_E f \, \mathrm{d}\mu = 0$ even if $f \equiv \infty$, and

(f) If $f \geqslant 0$ then $\int_E f \, \mathrm{d}\mu = \int_X \chi_E f \, \mathrm{d}\mu$.

(1.25) Let $s, t \geqslant 0$ be measurable simple functions on X. For $E \in \mathfrak{M}$ define $\varphi(E) := \int_E s \, \mathrm{d}\mu$. Then

$$\int_X (s + t) \, \mathrm{d}\mu = \int_X s \, \mathrm{d}\mu + \int_X t \, \mathrm{d}\mu,$$

and φ is a measure on $\mathfrak{M}$. *Rewrite* $\mu(A_i \cap E)$ *as the sum of a bunch of* μ*'s using its countable additivity.*

(1.26) **Lebesgue's MCT**. If $\{f_n\}$ is a sequence of measurable functions on X such that $0 \leqslant f_1 \leqslant \ldots \leqslant \infty$ and $f_n \to f$ pointwise, then f is measurable with $\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu$.

(1.27) If $f_n : X \to [0, \infty]$ is measurable and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then

$$\int_X f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \, \mathrm{d}\mu.$$

Consider doing this inductively: for f_1, f_2, use 1.17 and MCT; then repeat the process.

(1.28) **Fatou's Lemma**. If $f_n : X \to [0, \infty]$ is measurable, then

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu.$$

Define $g_k := \inf_{i \geqslant k} f_i$ and notice that $g_k \leqslant f_k$. 1.14 implies $g_k \to \liminf f_n$ and the claim follows from MCT.

(1.29) If $f : X \to [0, \infty]$ is measurable and $\varphi(E) := \int_E f \, \mathrm{d}\mu$ is a measure on $\mathfrak{M}$ with $\int_X g \, \mathrm{d}\varphi = \int_X fg \, \mathrm{d}\mu$. *Write E as the disjoint union of $E_1, E_2, \ldots$ and use 1.25 and 1.27 for the first claim. Then use 1.17 and MCT for the second claim.*

(1.30) $L^1(\mu)$ is the collection of all Lebesgue integrable and complex measurable functions f on X.

(1.31) If $f = u + iv$ where u, v are real measurable and if $f \in L^1(\mu)$, define

$$\int_E f \, \mathrm{d}\mu = \int_E u^+ \, \mathrm{d}\mu - \int_E u^- \, \mathrm{d}\mu + i \int_E v^+ \, \mathrm{d}\mu - i \int_E v^= \, \mathrm{d}\mu.$$

(1.32) Lebesgue integrals of L^1 functions are linear: if $f, g \in L^1(\mu)$ then $\alpha f + \beta g \in L^1(\mu)$ with

$$\int_X (\alpha f + \beta g) \, \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu + \beta \int_X g \, \mathrm{d}\mu.$$

Use 1.24 and 1.27 to bound its L^1 norm and use f^+, f^-, etc. to prove linearity of $f + g$.

(1.33) If $f \in L^1(\mu)$ then $\left| \int_X f \, \mathrm{d}\mu \right| \leqslant \int_X |f| \, \mathrm{d}\mu$. *Write $z = \int_X f \, \mathrm{d}\mu$ and choose $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $\alpha z = |z|$. Let $u = \mathfrak{Re}(\alpha f)$ so $u \leqslant |\alpha f| = |f|$. Then*

$$\left| \int_X f \, \mathrm{d}\mu \right| = \alpha \int_X f \, \mathrm{d}\mu = \int_X \alpha f \, \mathrm{d}\mu = \int_X u \, \mathrm{d}\mu \leqslant \int_X |f| \, \mathrm{d}\mu.$$

(1.34) **Lebesgue's DCT**. If $\{f_n\}$ is a sequence of complex measurable functions on X that converges pointwise to f and if there exists $g \in L^1(\mu)$ that bounds every f_n absolutely (i.e., $|f_n(x)| \leqslant g(x) \ \forall x, n$), then $f \in L^1(\mu)$ with

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu.$$

Apply Fatou's lemma to $2g - |f_n - f|$.

(1.36) Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\mathfrak{M}^*$ be the collection of all $E \subset X$ such that $A \subset E \subset B$ for some $A, B \in \mathfrak{M}$ with $\mu(B - A) = 0$. Also define $\mu'(E) = \mu(A)$. Then $(X, \mathfrak{M}^*, \mu')$ is a measure space.

(1.38) If $\{f_n\}$ are complex measurable functions defined a.e. on X with

$$\sum_{n=1}^{\infty} \int_X |f_n| \, \mathrm{d}\mu < \infty,$$

then $f(x) := \sum_{n=1}^{\infty} f_n(x)$ converges a.e., $f \in L^1(\mu)$, and $\int_X f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \, \mathrm{d}\mu$.

If f_n is defined on S_n then f is defined on $S = \bigcap_{n=1}^{\infty} S_n$ and $\mu(S^c) = 0$. Further define $E := \{x \in S : \sum |f_n(x)| < \infty\}$. Use DCT on E and the claim follows since $\mu(E^c) = \mu(S - E) + \mu(S^c) = 0$.

(1.39) (a) Suppose $f : X \to [0,\infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f \, \mathrm{d}\mu = 0$. Then $f = 0$ a.e. on E. *Define $A_n := \{x \in E : f(x) > 1/n\}$ and notice that*

$$\frac{\mu(A_n)}{n} \leqslant \int_{A_n} f \, \mathrm{d}\mu \leqslant \int_E f \, \mathrm{d}\mu = 0.$$

(b) Suppose $f \in L^1(\mu)$ and $\int_E f \, \mathrm{d}\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X. *Let $f = u + iv$ and let $E = \{x : u(x) \geqslant 0\}$. Since $\mathfrak{Re} \int_E f \, \mathrm{d}\mu = \int_E u^+ \, \mathrm{d}\mu = 0$, by (a) $\mu^+ \equiv 0$ a.e. Likewise for the other parts.*

(c) Suppose $f \in L^1(\mu)$ and $\left| \int_X f \, \mathrm{d}\mu \right| = \int_X |f| \, \mathrm{d}\mu$. Then there exists a constant α with $\alpha f = |f|$ a.e. on X.

Refer to proof of 1.33: we need $u = \mathfrak{Re}(\alpha f) = |f|$ a.e.

(1.40) If $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in $\mathbb{C}$, and

$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu$$

is in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$, then $f(x) \in S$ for almost all $x \in X$. *Consider closed (or open) circular discs $B(\alpha, r)$'s in S^c and define $E = f^{-1}(B(\alpha, r))$. If $\mu(E) > 0$ then*

$$|A_E(f) - \alpha| = \frac{1}{\mu(E)} \left| \int_E f - \alpha \, \mathrm{d}\mu \right| \leqslant \frac{1}{\mu(E)} \int_E |f - \alpha| \, \mathrm{d}\mu \leqslant r$$

(since $|f(x) - \alpha| \leqslant r$ for $x \in E$), contradiction.

(1.41) If $\{E_k\}$ a sequence of measurable sets in X satisfy $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, then almost all $x \in X$ lie in finitely many of the sets E_k. *Let A be "opposite" set and define $g(x) := \sum_{k=1}^{\infty} \chi_{E_k}(x)$. Integrating g over X and using 1.27. the integral is precisely $\sum_{k=1}^{\infty} \mu(E_k)$, so $g(x)$ must be $< \infty$ a.e.*