Rudin RCA Chapter 2 Exercises

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Problem 1

Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbb{R}$ and consider the following statements:

(a) If f_1, f_2 are upper semicontinuous, then so is $f_1 + f_2$.

(b) If f_1, f_2 are lower semicontinuous, then so is $f_1 + f_2$.

(c) If each f_n is upper semicontinuous, then so is $\sum_{n=1}^{\infty} f_n$.

(d) If each f_n is lower semicontinuous, then so is $\sum_{n=1}^{\infty} f_n$.

Show that three of these are true and one false. What happens if the word "nonnegative" is omitted? What if we replace $\mathbb{R}$ by a general topological space?

Proof. (a) True, and we will use neither the nonnegativity assumption nor $\mathbb{R}$'s topological structure:

$$\begin{aligned}\{x : f_1(x) + f_2(x) < \alpha\} &= \{x : f_1(x) < \alpha - f_2(x)\} \\ &= \bigcup_{r\in\mathbb{R}} \{x : f_1(x) < r < \alpha - f_2(x)\} \\ &= \bigcup_{q\in\mathbb{Q}} \{x : f_1(x) < q < \alpha - f_2(x)\} \\ &= \bigcup_{q\in\mathbb{Q}} \{x : f_1(x) < q\} \cap \{x : f_2(x) < \alpha - q\}.\end{aligned}$$

The second = is because $\mathbb{Q}$ is dense in $\mathbb{R}$; each $\{x : f_1(x) < q\} \cap \{x : f_2(x) < \alpha - q\}$ is the intersection of two open sets and is thus open; and the last line represents a union of open sets and is therefore open.

(b) True, and the proof is analogous to that of (a).

(c) False: let $\{q_n\}_{n\geqslant 1}$ be an enumeration of the rationals and define f_n to be χ_{q_n}, i.e., 1 at the rational q_n and zero everywhere else. Then obviously each f_n is upper semicontinuous as the singleton $\{q_n\}$ is closed. It is also clear that the sum of all f_n's is $\chi_{\mathbb{Q}}$, which is not upper semicontinuous: $\{x : \sum f < 1\}$ is the set of irrationals which is not open in $\mathbb{R}$.

(d) True, but only if we keep the word "nonnegative": from (a) we know that each $g_n := f_1 + \ldots + f_n$ is upper semicontinuous, and if each f_n is nonnegative,

$$\sum f(x) > \alpha \iff g_n > \alpha \text{ for some } n$$

(and the second statement actually implies $g_m > \alpha$ for all $m \geqslant n$). Therefore

$$\{x : \sum f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : g_n(x) > \alpha\},$$

a (countable) union of open sets, proving the claim.

However, if the nonnegativity assumption were to be dropped, let us consider $h_n := -f_n$ where f_n is defined as in (c). Then $\sum h = -\chi_{\mathbb{Q}}$, and the set $\{x : \sum h > -1\}$ is again the set of irrationals which is not open in $\mathbb{R}$. □

Problem 2

Let f be an arbitrary complex function on $\mathbb{R}$ and define

$$\varphi(x) := \inf_{\delta>0} \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\}.$$

Prove that φ is upper semicontinuous, that f is continuous if and only if $\varphi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ. Formulate and prove an analogous statement for a general topological space.

Proof. We first provide a generalized statement:

> If X is a topological space and $f : X \to \mathbb{C}$ an arbitrary function, then the function
>
> $$\varphi(x) := \inf_{x \in U} \sup\{|f(s) - f(t)| : s, t \in U, U \text{ open}\}$$
>
> is upper semicontinuous, uniformly equals 0 if and only if f is continuous, and whose set of continuity points form a G_δ.

For the first claim, consider an arbitrary $\alpha \in \mathbb{R}$ and the corresponding set $S = \{x : \varphi(x) < \alpha\}$. If $x \in S$ then, by definition of infimum, there exists an open set U containing x such that $\sup\{|f(s), f(t)| : s, t \in U\} < \alpha$. Since this holds for any $x \in S$ we conclude that S is open and hence φ is upper semicontinuous.
Both directions of the statement (f continuous $\Leftrightarrow \varphi \equiv 0$) follow directly from the definition.
Finally, recall that G_δ contains sets that are countable intersection of open sets, and our set of continuity points, from the previous part, is equal to $\bigcap_{n=1}^{\infty} \{x \in X : \varphi(x) < 1/n\}$. □

Problem 3

Let X be a metric space with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A, B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Proof. Let $\epsilon > 0$ be given. We claim that the corresponding δ for the uniform continuity condition is $\epsilon/2$. Indeed, let $x, y \in X$ be arbitrarily chosen with $\rho(x,y) < \epsilon/2$. By definition there exists $e \in E$ with $\rho(x,e) < \rho_E(x) + \epsilon/2$. Hence

$$\rho_E(y) \leqslant \rho(e,y) \leqslant \rho(e,x) + \rho(x,y) < \rho(x,y) + \rho_E(x) + \epsilon/2$$

and so $\rho_E(y) - \rho_E(x) < \rho(x,y) + \epsilon/2$. Likewise one can show $\rho_E(x) - \rho_E(y) < \rho(x,y) + \epsilon/2$. The RHS is precisely ϵ, so this concludes the proof.

For the function f, notice that it is uniformly 1 if $x \in B$ (so $\rho_B(x)$ vanishes) and uniformly 0 if $x \in A$ (so $\rho_A(x)$ vanishes). Therefore this is Urysohn's lemma applied to the open set $X - A$ and the closed set B. □

Problem 4

Examine the proof of the Riesz theorem and prove the following two statements:

(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$ where V_1, V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1, E_2 are not in $\mathfrak{M}$.

(b) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \ldots$ where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

Proof. (a) If $\mu(E_i) = \infty$ then we are already done. Otherwise, assume both have finite measure.

Using (Step I) we already know $\mu(E_1 \cup E_2) \leqslant \mu(E_1) + \mu(E_2)$, so it suffices to show the other direction. Recall that μ is defined as

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}.$$

Now let S be an arbitrary open set containing $E_1 \cup E_2$. By definition $\mu(S) \geqslant \mu(E_1 \cap E_2)$. On the other hand, consider $S \cap V_1$ and $S \cap V_2$, two disjoint open sets. (Step III) and (Step IV) together imply

$$\mu((S \cap V_1) \cup (S \cap V_2)) = \mu(S \cap V_1) + \mu(S \cap V_2).$$

Clearly $E_i \subset (S \cap V_i)$, so

$$\mu(S) \geqslant \mu((S \cap V_1) \cup (S \cap V_2)) = \mu(S \cap V_1) + \mu(S \cap V_2) \geqslant \mu(E_1) + \mu(E_2).$$

Taking infimum on both sides, we obtain $\mu(E) \geqslant \mu(E_1) + \mu(E_2)$, and thus the equation holds.

(b) Since $E \in \mathfrak{M}_F$, by (Step V) there exist compact K_1 and open V_1 with $K_1 \subset E \subset V_1$ and $\mu(V_1 - K_1) < 1$. In particular, $\mu(E - K_1) < 1$. Define $E_1 := E - K_1$ which, by (Step IV), is also a member of $\mathfrak{M}_F$. Then we can find $K_2 \subset E_1 \subset V_2$ with $\mu(E_1 - K_2) < \mu(V_2 - K_2) < 1/2$ and iterate the process inductively. It follows that

$$\mu(E - \bigcup_{i=1}^{n} K_i) < 1/n \implies \mu(E - \bigcup_{i=1}^{\infty} K_i) = 0.$$

Define this set of measure zero to be N and we are done with the proof.

□

Problem 5

Let E be the middle thirds Cantor set. Show that $m(E) = 0$ even though $|E| = |\mathbb{R}|$.

Proof. Let E_n be the n^{th} iteration in the construction of the Cantor set (i.e., E_0 consists of $[0, 1]$, E_1 of $[0, 1/3]$ and $[2/3, 1]$, and so on). It follows that $m(E_n) = (2/3)^n$ whereas

$$E = \bigcap_{n=0}^{\infty} E_n \implies \mu(E) \leqslant \mu(E_n) \text{ for all } n \implies \mu(E) = 0.$$

On the other hand, there exists a bijection between elements of E and binary decimals in $[0, 1]$ which, in turn, can be bijectively mapped into $\mathbb{R}$. □

Problem 6

Construct a totally disconnected compact set $K \subset \mathbb{R}$ such that $m(K) > 0$. If v is lower semicontinuous and $v \leqslant \chi_K$, show that actually $v \leqslant 0$. Hence χ_K cannot be approximated from below by lower continuous functions in the sense of the Vitali-Carathéodory theorem.

Proof. Yet another classic 425a question... Consider a modified Cantor set such that, in the n^{th} iteration, we remove line segments with length $1/2^{2n}$ from the middle of each segment from the previous iteration. (The initial segment is still $[0, 1]$, the first gives $[0, 3/8]$ and $[5/8, 1]$, and the third iteration removes two segments of length $1/16$, and so on.) A simple calculation shows that

$$\mu(E_1) = 1 - \frac{1}{4},\ \mu(E_2) = \mu(E_1) - \frac{2}{16},\ \dots,\ \lim_{n\to\infty} \mu(E_n) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \frac{1}{2}.$$

Since E is obviously bounded and closed (intersection of closed sets) it is compact, satisfying our requirements. If $v \leqslant \chi_K$ then $\{x : v(x) > 0\}$ is open, and the only possibility is if it is $\varnothing$, i.e., $v \leqslant 0$. □

Problem 7

If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$ such that $m(E) = \epsilon$.

Solution. The complement of the set in problem 6 provides a dense open set with measure $1 - 1/2 = 1/2$. All we need to do is to find an appropriate number $k = k(\epsilon)$ such that

$$\frac{1}{k} + \frac{2}{k^2} + \dots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{k^n} = \frac{1}{k} \cdot \frac{k}{k-2} = \frac{1}{k-2} = \epsilon.$$

It follows that $k = 1/\epsilon + 2$, and we simply need to remove the middle k^{th} of each segment in each iteration and take the complement of the entire set. This modified Cantor set is closed and nowhere dense, so its complement is open and dense. □

Problem 8

Construct a Borel set $E \subset \mathbb{R}$ with $0 < m(E \cap I) < m(I)$ for every nonempty segment I. Is it possible to have $m(E) < \infty$ for such a set?

Solution. Let $\{q_n\}_{n\geqslant 1}$ be an enumeration of the rationals. Let ϵ be any positive number. Define open sets $\{V_n\}$ by $V_1 = (q_1 - \epsilon/3, q_1 + \epsilon/3)$ and $V_n = (q_n - \epsilon/3^n, q_n + \epsilon/3^n)$ similarly. Define

$$W_n = V_n - \bigcup_{i=n+1}^{\infty} V_i$$

so that W_n's are disjoint. Notice that

$$m(W_n) \geqslant m(V_n) - m(\bigcup_{i=n+1}^{\infty} V_i) \geqslant m(V_n) - \sum_{i=n+1}^{\infty} m(V_i) = \epsilon/3^n > 0.$$

Now for each W_n, pick Borel set $E_n \subset W_n$ with $0 < m(E_n) < m(W_n)$ and define $E := \bigcup_{n=1}^{\infty} E_n$. It follows immediately that $0 < m(E \cap W_n) = m(E_n) < m(W_n)$ for all n.

Furthermore, for an arbitrary nonempty segment I, since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists k large enough such that $V_k \subset I$, and so $W_k \subset I$. This implies $0 < m(E \cap I) < m(I)$, whereas $m(E) \leqslant m(\bigcup_{n=1}^{\infty} W_n) \leqslant m(\bigcup_{n=1}^{\infty} V_n) < \infty$. □

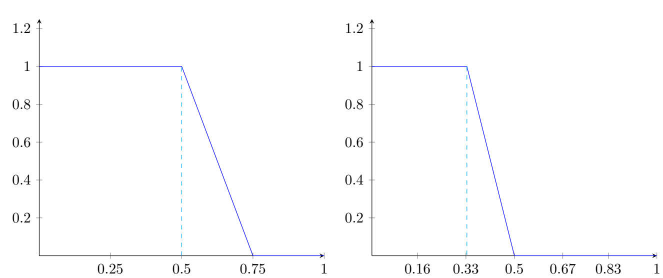
Problem 9

Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \leqslant f_n \leqslant 1$, $\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = 0$, but $\{f_n(x)\}$ converges for no $x \in [0,1]$.

Solution. This idea is borrowed from Gary Rosen's 407. First define a sequence of functions $\{g_n\}$ by

$$\begin{aligned}
&g_1 = \chi_{[0,1]}\\
&g_2 = \chi_{[0,1/2]},\ g_3 = \chi_{[1/2,1]}\\
&g_4 = \chi_{[0,1/3]},\ g_5 = \chi_{[1/3,2/3]},\ g_6 = \chi_{[2/3,1]}\\
&g_7 = \chi_{[0,1/4]},\ \cdots
\end{aligned}$$

Clearly the integral of functions on the n^{th} line is $1/n$ which tends to 0, whereas each $x \in [0,1]$ is always contained in some interval of length $1/n$, and so no $\{f_n(x)\}$ converges as the value 1 keeps popping up once in a while. Now we just need to modify g_n's to make them continuous. Interpolation can be a method; alternatively, consider the following that redefines g_2 and g_4 to be f_2 and f_4:



An explicit formula can be derived but it serves little purpose here. It is clear that the integral still converges to 0, that each function is continuous, and that $\{f_n\}$ is nowhere pointwise convergent. This proves the claim. $\square$

Problem 10

If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $0 \leqslant f_n \leqslant 1$ and such that $f_n(x) \to 0$ as $n \to \infty$, prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = 0$$

without using measure theory.

Proof. See this. *This is not really an exercise; it is only meant to show the elegance of Lebesgue integrals.* $\square$

Problem 11

Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint: let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$; show that every open set V which contains K also contains some K_α. Regularity of μ is needed; compare Ex.18. Show that K^c is the largest open set in X whose measure is 0.*

Proof. Define K as instructed by the hint. It follows that K is a closed subset of compact sets K_α so it itself is compact. Let $V \subset X$ be any open set containing K, and consider V^c which is compact (V^c closed and X compact). It follows that $\{K_\alpha^c \cap V^c\}$ covers V^c, and compactness reduces this cover to a finite subcover:

$$V^c = \bigcup_{i=1}^m (K_{\alpha_i}^c \cap V^c) = V^c \cap \bigcup_{i=1}^m K_{\alpha_i}^c.$$

Notice that $\mu(K_\alpha^c) = \mu(X) - \mu(K_\alpha) = 0$, so $\mu(V^c) = 0$. Hence every open supset of K has measure 1, and since μ is regular (in particular outer regular), $\mu(K) = 1$.

Finally, any $H \subsetneq K$ with $\mu(H) = 1$ violates the construction of K (if so, K needs to be a subset of U). $\square$

Problem 12

Show that every compact subset of $\mathbb{R}$ is the support of a Borel measure.

Proof. Let $K \subset K$ be a compact set and let m be the Lebesgue measure. Consider

$$\Lambda f = \int_K f\, \mathrm{d}m \qquad f \in C_c(\mathbb{R})$$

which, by Riesz RT (2.14), is equal to $\Lambda f = \int_K f\, \mathrm{d}m = \int_{\mathbb{R}} f\, \mathrm{d}\mu$ for some measure μ corresponding to $\mathfrak{M}$ containing all Borel sets in $\mathbb{R}$. Using 2.14 Step 2,

$$\mu(K) = \inf\{\Lambda f : K \prec f\} = \int_K 1\, \mathrm{d}m = m(K)$$

and, using the direct definition of μ in 2.14,

$$\mu(X) = \sup\{\Lambda f : f \prec X\} = \int_K 1\, \mathrm{d}m = m(K).$$

It follows that $m(K) = \mu(K) = \mu(X)$. Therefore $\mu(K^c) = 0$, and

$$\mu(E) = \mu(E \cap K) + \mu(E \cap K^c) = \mu(E \cap K),$$

proving that μ has support K.

□

Problem 13

Is it true that every compact subset of $\mathbb{R}$ is the support of a continuous function? If not, can you describe the class of all compact sets in $\mathbb{R}$ which are supports of continuous functions? Is your description valid in other topological spaces?

Solution. First question: no. A singleton in $\mathbb{R}$ is compact but clearly no continuous function can have a singleton as its support.

For part two, the class of all such sets is precisely the collection of those that are the closure of their interiors.

Part 3 [??] □

Problem 14

Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g, h such that $g(x) = h(x)$ a.e.$[m]$ and $g(x) \leqslant f(x) \leqslant h(x)$ for every $x \in \mathbb{R}^k$.

Proof. First assume $0 \leqslant f < 1$. Similar to Rudin's proof of Lusin's Theorem (2.24), we let $\{s_n\}$ be a sequence of simple functions converging pointwise to f. Define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ so that $2^n t_n$ is the characteristic function of $T_n \subset A$ and

$$f(x) = \sum_{n=1}^{\infty} t_n(x) \quad x \in \mathbb{R}^k.$$

Using 2.20(b), for each T_n there correspond $F_n \in F_\sigma$ and $G_n \in G_\delta$ with

$$F_n \subset T_n \subset G_n \qquad m(G_n - F_n) = 0.$$

Now we define

$$g := \sum_{n=1}^{\infty} 2^{-n} \chi_{F_n} \qquad \text{and} \qquad h := \sum_{n=1}^{\infty} 2^{-n} \chi_{G_n}.$$

It is immediate that $g \leqslant f \leqslant h$ and that, for each n, $t_n(x)$ agree a.e.$[m]$ with $2^{-n}\chi_{F_n}$ and $2^{-n}\chi_{G_n}$. Therefore $f = g = h$ a.e.$[m]$.

The case for bounded f follow immediately, and for unbounded f, we again refer to 2.24's proof. Define $B_n := \{x : |f(x)| < n\}$ so $f\chi_{B_n} \to f$ as $n \to \infty$. For each n we construct corresponding g_n, h_n as above and define

$$g := \limsup_{n \to \infty} g_n \qquad h := \liminf_{n \to \infty} h_n.$$

It follows that $g \leqslant f \leqslant h$ and $g(x) = h(x)$ a.e.$[m]$, completing the proof. □

Problem 15

It is easy to guess the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx \qquad \text{and} \qquad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx$$

as $n \to \infty$. Prove that your guesses are correct.

Proof. For the first integral, we define $f_n(x) := (1 - x/n)^n e^{x/2}$ on $[0, n]$ and zero on (n, ∞). Notice that

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n-x}{n}\right)^n = \lim_{n\to\infty} \left(\frac{1}{n/(n-x)}\right)^n$$
$$= \lim_{n\to\infty} \left(\frac{1}{1 + x/(n-x)}\right)^n = \frac{1}{e^x} = e^{-x}$$

and that $(1 - x/n)^n \leqslant 1$ on $[0, n]$, which implies $|f_n(x)| \leqslant f(x)$ for all n, all x. We can then invoke the Lebesgue DCT (1.34) and conclude

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx = \int_0^\infty e^{-x} e^{x/2}\, dx = 2.$$

Likewise, for the second integral, defining

$$g_n(x) := \chi_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x}$$

gives $\lim_{n\to\infty} g_n(x) = e^x e^{-2x}$ and so

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = \int_0^\infty e^{-x}\, dx = 1.$$ □

Problem 16

Why is $m(Y) = 0$ in the proof of Theorem 2.20(e)?

Proof. It suffices to prove that $m(\mathbb{R}) = 0$ when we view $\mathbb{R}$ as a subspace of $\mathbb{R}^2$ (the general case $\mathbb{R}^m \subset \mathbb{R}^n$ follows inductively). Let m be the Lebesgue measure on $\mathbb{R}^2$, let $\{z_n\}_{n\geqslant 1}$ be an enumeration of the integers and let $\epsilon > 0$ be given, and let $\{I_n\}_{n\geqslant 1}$ be the collection of intervals $[z_n, z_n + 1)$. Clearly the union of I_n is $\mathbb{R}$. Now we cover I_1 using

$$W_1 := [z_1, z_1 + 1] \times [-\epsilon/4, \epsilon/4] \implies m(W_1) = \frac{\epsilon}{2}.$$

Inductively, we can cover I_n using

$$W_n := [z_n, z_n + 1] \times [-\epsilon/2^{n+1}, \epsilon/2^{n+1}] \implies m(W_n) = \frac{\epsilon}{2^n}.$$

It follows immediately that $\bigcup_{n=1}^{\infty} W_n$ covers $\mathbb{R}$ with total measure $= \epsilon$. Since ϵ is arbitrary, $m(\mathbb{R}) = 0$. □

Problem 17

Define the distance between (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1, y_2| \text{ if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.$$

Show that this is indeed a metric and that the resulting metric space X is locally compact. If $f \in C_c(X)$, let $x_1, \ldots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x [!]), and define

$$\Lambda f = \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x_i, y)\, \mathrm{d}y.$$

Let μ be the measure associated with this Λ by Theorem 2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

Proof. We first show that $d((x_1, y_1), (x_2, y_2))$ [d denoting the distance metric] as defined in the question is a metric; everything besides triangle inequality is clear. Let $(x_1, y_1), (x_2, y_2)$ be given. If $x_1 = x_2$ then $d((x_1, y_1), (x_2, y_2)) = |y_1 - y_2|$ and

$$|y_1 - y_2| \leqslant |y_1 - y_3| + |y_3 - y_2|$$

for any y_3, so regardless of whether $x_3 = x_1 = x_2$ or not (in which case the RHS is even larger), the inequality holds. For $x_1 \neq x_2$, $d((x_1 y_1), (x_2, y_2)) = 1 + |y_1 - y_2|$. Note that

$$1 + |y_1 - y_2| \leqslant 1 + |y_1 - y_3| + |y_3 - y_2|,$$

and for any x_3, we have $(x_1 \neq x_3) \vee (x_2 \neq x_3)$, so either $d((x_1, y_1), (x_3 y_3))$ contains that 1 or $d((x_2, y_2), (x_3, y_3))$ does (or most likely both!). Either way the triangle inequality holds.

Now we show that X is locally compact. Consider an arbitrary (x, y). Notice that if we set $0 < r < 1$, then all points (x', y') in $B((x, y), r)$ [open ball centered at (x, y) with radius r] must have $x' = x$. That is,

$$B((x, y), r) = \{(x', y') : x' = x, |y' - y| < r\},$$

and the closure is obtained by changing $<$ to $\leqslant$. Since $[y - r, y + r]$ is a compact subset of $\mathbb{R}^2$, we conclude that $(\mathbb{R}^2, d)$ is locally compact.

Now we show that there are only finitely many x_n's for which $f(x, y) \neq 0$. Let $f \in C_c(X)$ with (compact) support K. For $0 < r < 1$, there exists a finite cover of K using r-balls:

$$K \subset \bigcup_{i=1}^{n} B((x_i, y_i), r).$$

As shown above, all points in a r-ball have the same x-coordinate, so this shows K can have only finitely many distinct x_i's.

Finally we deal with E. It is clear that Λ is a positive linear functional and thus the Riesz RT (2.14) applies and gives us a measure μ. To show $\mu(E) = \infty$, it suffices to show that the set $V := \{(x, y) : y \in (-\epsilon, \epsilon)\}$ has infinite measure for all $\epsilon > 0$. For simplicity we shall limit $\epsilon < 1$. Note that V is an open set. Now define

$$E_n := \bigcup_{i=1}^{n} \overline{B((x_i, 0), \epsilon/2)},$$

where x_i's are arbitrarily chosen, subject only to the conditions that the closed balls are mutually disjoint (i.e., x_i's are $> \epsilon$ from each other). It follows that E_n is a finite union of compact sets and is therefore compact. Now we have $E_n \subset E \subset V$, so Urysohn's lemma (2.12) provides the existence of $f \in C_c(X)$ with $F \prec f \prec V$. In particular, $f(x) = 1$ for all $x \in F$. On the other hand, Λf is the sum of *at least* n terms (there may be more x_i's but we don't care), each corresponding to a closed ball, whose integral evaluates to ϵ:

$$\Lambda f = \sum_{i=1}^{m} \int_{-\infty}^{\infty} f(x_i, y)\,\mathrm{d}y \geqslant \sum_{i=1}^{n} \int_{-\infty}^{\infty} f(x_i, y)\,\mathrm{d}y \geqslant n \int_{-\infty}^{\infty} \chi_{(-\epsilon/2, \epsilon/2)}\,\mathrm{d}y = n\epsilon.$$

This means

$$\mu(V) \geqslant \Lambda f \geqslant n\epsilon.$$

Fixing this ϵ and letting $n \to \infty$, we see $\mu(V) = \infty$. Since ϵ was chosen arbitrarily in the first place, we conclude that $\mu(E) = \infty$.

Finally, let $K \subset E$ be a compact set. Notice that K must be finite (or a sequence may have term-wise distance 1 and never admit a convergent subsequence). Writing

$$K = \{(x_1, 0), \ldots, (x_k, 0)\}$$

and defining

$$U_n := \bigcup_{i=1}^{k} B((x_i, 0), 1/k),$$

we see that as $n \to \infty$, $\mu(U_n) \to 0$, so $K \subset U_n$ implies $\mu(K) = 0$, as claimed. □

Problem 19

Go through the proof of Theorem 2.14, assuming X to be compact (or even compact metric) rather than just locally compact, and see what simplifications you can find.

Solution. If X is compact then Λf is automatically finite, so there is no need to check statements like if $\mu(E) = \infty$. In particular, $\mathfrak{M}_F = \mathfrak{M}$, making Step VIII redundant.

Problem 20

Find continuous functions $f_n : [0,1] \to [0, \infty)$ such that $f_n(x) \to 0$ for all $x \in [0,1]$ as $n \to \infty$, $\int_0^1 f_n(x)\,\mathrm{d}x \to 0$, but $\sup f_n$ is not in L^1. *This shows that the conclusion of the Lebesgue DCT may hold even when part of its hypothesis is violated.*

Solution. Idea comes from Pugh's *Real Mathematical Analysis*, §4.1, in which he presents the *growing steeple*.

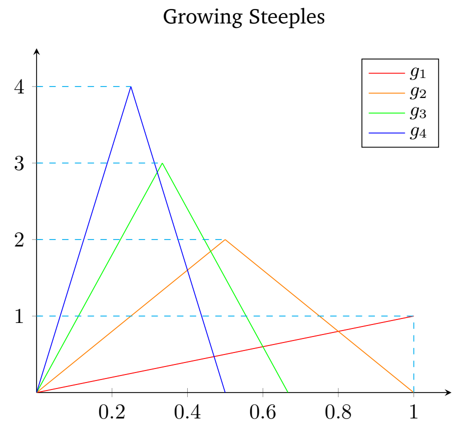
$$g_n(x) = \begin{cases} n^2 x & 0 \leqslant x \leqslant 1/n \\ 2n - n^2 x & 1/n \leqslant x \leqslant 2/n \\ 0 & 2/n \leqslant x \leqslant 1. \end{cases}$$

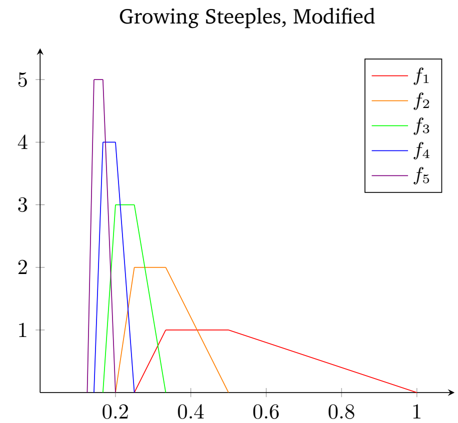
(See figures below.) Unfortunately, calculating the supremum of these functions is a pain, and the integrals do not converge to 0. We can, however, circumvent that issue by "flattening the peak" to get nicer numbers and making the steeples thinner by using $1/n^2$ rather than $1/n$ (so it outgrows the height of the peak, and this is done by constructing $1/n - 1/(n+1)$). For $n \geqslant 1$, define

$$f_n(x) := \begin{cases} n(n+2)(n+3) & 1/(n+3) \leqslant x \leqslant 1/(n+2) \\ n & 1/(n+2) \leqslant x \leqslant 1/(n+1) \\ n(n+1) - n^2(n+1)x & 1/(n+1) \leqslant x \leqslant 1/n \\ 0 & \text{otherwise.} \end{cases}$$

Despite its messy appearance, each f_n is in fact the piecewise function connecting the following points:

$$(0,0) \leftrightarrow (1/(n+3),0) \leftrightarrow (1/(n+2),n) \leftrightarrow (1/(n+1),n) \leftrightarrow (1/n,0) \leftrightarrow (1,0).$$





It is clear that the integral converges to 0:

$$\begin{aligned} \int_0^1 f_n(x)\,\mathrm{d}x &= \frac{1}{2(n+2)(n+3)} + \frac{n}{(n+1)(n+2)} + \frac{1}{2n(n+1)} \\ &\leqslant \frac{n}{(n+1)(n+2)} + \frac{1}{n(n+1)} \leqslant \frac{n+1}{n(n+1)} = \frac{1}{n}, \end{aligned}$$

and it is clear that each f_n converges pointwise to the zero function: for any x, pick n large enough such that $1/n < x$. Then $f_m(x) = 0$ for all $m \geqslant n$. Now it remains to show that $\sup f_n \notin L^1$. Define $f := \sup f_n$ for convenience. We now provide an estimate for $\|f\|_{L^1}$ using the "peaks" only (hence $>$ not $\geqslant$):

$$\begin{aligned} \|f\|_{L^1} &= \int_0^1 f(x)\,\mathrm{d}x > \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/(n+2)} f(x)\,\mathrm{d}x \\ &> \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} \sim \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

This completes the construction of $\{f_n\}$[!] □

Problem 21

If X is compact and $f : X \to (-\infty, \infty)$ upper semicontinuous, prove that f attains its maximum at some point of X.

Proof. The sets $\{x : f(x) < \alpha\}$ are open by assumption. Using the compactness of X, there exists a finite set $\{\alpha_1, ..., \alpha_n\}$ such that

$$\bigcup_{i=1}^{n} \{x : f(x) < \alpha_i\}$$

covers X. It follows that f is bounded and we can define $\alpha := \sup\{f(x) : x \in X\}$ (which is finite). If no $x \in X$ gives $f(x) = \alpha$, then there exists a sequence $\{\beta_n\}$ approaching α from below. Then

$$\bigcup_{n=1}^{\infty} \{x : f(x) < \beta_n\}$$

covers X but does not have a finite subcover, contradiction. Therefore f attains maximum inside X. □

Problem 22

Suppose that X is a metric space, with metric d, and that $f : X \to [0, \infty]$ is lower semicontinuous, $f(p) < \infty$ for at least one $p \in X$. For $n \geqslant 1$, $x \in X$, define

$$g_n(x) := \inf\{f(p) + nd(x, p) : p \in X\}.$$

Prove that

(a) $|g_n(x) - g_n(y)| \leqslant nd(x, y)$,

(b) $0 \leqslant g_1 \leqslant ... \leqslant f$, and

(c) $g_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions.

Proof. (a) Let p be such that $f(p) < \infty$. WLOG assume $g_n(x) > g_n(y)$. Then

$$nd(x, y) \geqslant n(d(x, p) - d(y, p)) = nd(x, p) + f(p) - nd(y, p) - f(p) \geqslant g_n(x) - g_n(y).$$

(b) Since $f(p) \geqslant 0$ and $d(x, p) \geqslant 0$, $g_1 \geqslant 0$. Note that

$$f(p) + (n-1)d(x, p) \leqslant f(p) + nd(x, p) \implies g_{n-1} \leqslant g_n.$$

Finally, if $f = \infty$ then $f \geqslant g_n$ holds trivially; if $f \neq \infty$, letting $x = p$ gives $f(x) + nd(x, x) = f(x)$, so $g_n(x) \leqslant f(x)$ by definition of imfimum.

(c) If $f = \infty$ then $nd(x, p) \to \infty$, and the claim holds. If $f \neq \infty$, as $n \to \infty$, the infimum is obtained by letting $p = x$, which gives $g_n(x) = g(x)$ for all n, and convergence follows trivially.

□

Problem 23

Suppose V is open in $\mathbb{R}^k$ and μ is a finite Borel measure on $\mathbb{R}^k$. Is the function that sends x to $\mu(V + x)$ necessarily continuous? lower semicontinuous? upper semicontinuous?

Solution. μ need not to be continuous; in particular it need not to be upper semicontinuous. Consider $x_0 \in \mathbb{R}^k$ and a measure $\mu = \mu_{x_0}$ defined by

$$\mu(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E. \end{cases}$$

It is easy to verify that this μ is countably additive for disjoint sets and is therefore a measure. Now define $V := B(0, r)$ and consider the set $\{x : \mu(V) < 1\} = \{x : 0 \notin B(0, r) + x\}$. Notice that this set is the complement of $B(0, r)$ and is therefore closed, showing that $x \mapsto \mu(V + x)$ is not upper semicontinuous.

To show that the mapping is lower semicontinuous, let α be given and consider $\{x : \mu(V + x) > \alpha\}$. If this set is not open, then for some element x, we can construct a sequence $\{x_n\} \to x$ such that $\mu(V + x_n) \leqslant \alpha$ for all x_n. Note that as $V + x_n$ approaches $V + x$, if $y \in V + x$ then $y \in V + x_n$ for all sufficiently large n's (i.e., for all but finitely many n). This means

$$\liminf_{n\to\infty} \chi_{V+x_n} \geqslant \chi_{V+x}.$$

Using Fatou's lemma, we have

$$\int_{\mathbb{R}^k} \chi_{V+x}\, \mathrm{d}\mu \leqslant \int_{\mathbb{R}^k} \liminf_{n\to\infty} \chi_{V+x_n}\, \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^k} \chi_{V+x_n}\, \mathrm{d}\mu,$$

so $\mu(V + x) \leqslant \liminf \mu(V + x_n)$, and $\mu(V + x) > \alpha$ is clearly a contradiction. □

Problem 24

A step function is a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}$. Assuming $f \in L^1(\mathbb{R})$, prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)|\, \mathrm{d}x = 0.$$

Proof. We first show that simple functions can be approximated by step functions in $\|\cdot\|_{L^1}$. Since simple functions have finite range, they are linear combinations of characteristic functions, so it suffices to show that step functions are dense in the set of characteristic functions on measurable sets.

Let E be measurable with $m(E) < \infty$ and let $\epsilon > 0$ be given. By 2.17 there exists an open set V with $m(V - E) < \epsilon/2$. By 2.19 V is the countable union of disjoint open intervals I_n, so

$$m(U) = \sum_{n=1}^{\infty} m(I_n) \implies \sum_{n>k} m(I_n) < \frac{\epsilon}{2}$$

for sufficiently large k. Now we define the k^{th} truncated sum to be h. It follows that $h = \chi_E$ everywhere except on $(E - \bigcup_{n=1}^{k} I_n) \cup (\bigcup_{n=1}^{k} I_n - E)$. Since

$$m(V - E) < \frac{\epsilon}{2} \quad \text{and} \quad m(V - \bigcup_{n=1}^{k} I_n) < \frac{\epsilon}{2},$$

and $E \subset V$, $\bigcup_{n=1}^{k} I_n \subset V$, we conclude that

$$m(E - \bigcup_{n=1}^{k} I_n) + m(\bigcup_{n=1}^{k} I_n - E) \leqslant m(V - \bigcup_{n=1}^{k} I_n) + m(V - E) = \epsilon.$$

Therefore step functions are dense in simple functions w.r.t. L^1. It remains to show that simple functions are dense in L^1. Writing $f \in L^1(\mathbb{R})$ as $f^+ - f^-$ and using 1.17, there exist sequences of simple functions $\{g_n\}, \{h_m\}$ converging monotonically to f^+ and f^-, respectively. By the Lebesgue MCT (1.26),

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n \, \mathrm{d}x = \int_{\mathbb{R}} f^+ \, \mathrm{d}x \quad \text{and} \quad \lim_{m\to\infty} \int_{\mathbb{R}} h_m \, \mathrm{d}x = \int_{\mathbb{R}} f^- \, \mathrm{d}x.$$

In particular this means that $\{g_n - h_m\}$ approximates f w.r.t. $\|\cdot\|_{L^1}$. Therefore the set of simple functions is dense in $L^1(\mathbb{R})$, while the set of step functions is dense in that of simple functions. This concludes the proof. □

Problem 25

(a) Find the smallest constant c such that

$$\log(1+e^t) < c + t \qquad (0 < t < \infty).$$

(b) Does the following limit exist for every $f \in L^1$? If so, what is it?

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log(1+e^{nf(x)}) \, \mathrm{d}x.$$

Solution. (a) Since $\exp$ is monotone, $\log(1+e^t) < c+t$ if and only if $1 + e^t < e^c e^t \implies \log(1+e^{-t}) < c$. To minimize c we need to maximize the LHS, which approaches maximum as $t \to 0$, giving us $c = \log 2$.

(b) Let $X \subset [0,1]$ be the set on which $f \geqslant 0$. Using the previous part,

$$\lim_{n\to\infty} \frac{1}{n} \int_X \log(1+e^{nf(x)}) \, \mathrm{d}x < \lim_{n\to\infty} \frac{1}{n} \int_X \log 2 + nf(x) \, \mathrm{d}x = \int_X f(x) \, \mathrm{d}x,$$

whereas on $[0,1] - X$, $e^{nf(x)} \to 0$ so $\log(1+e^{nf(x)}) \to 0$. The Lebesgue DCT assures that

$$\lim_{n\to\infty} \frac{1}{n} \int_{[0,1]-X} \log(1+e^{nf(x)}) \, \mathrm{d}x = \frac{1}{n} \int 0 \, \mathrm{d}x = 0,$$

so the total limit is simply $\int_X f(x) \, \mathrm{d}x$.

□