Rudin RCA Ch2 Theorems

(2.3) Let X be a topological space (cf. 1.2).

- (a) $E \subset X$ is **closed** if E^c is open. ($\varnothing$ and X itself are closed.)
- (b) $\overline{E}$, the **closure** of E, is the smallest closed set in X containing E.
- (c) $K \subset X$ is compact if every open cover of K admits a finite subcover.
- (d) A **neighborhood** of $p \in X$ is any open subset containing p.
- (e) X is a **Hausdorff space** if, for distinct $p, q \in X$, p and q have disjoint neighborhoods.
- (f) X is **locally compact** if every $p \in X$ has a precompact neighborhood (compact closure).

(2.4) If $K \subset X$ is compact and $F \subset K$ closed, then F is compact.

(2.5) If X is Hausdorff, $K \subset X$ compact, and $p \in K^c$, then there exist disjoint open sets U and W containing K and p, respectively. *Use Hausdorff separation axiom on p and all $q \in K$ and use compactness.* In particular:

- (a) Compact subsets of Hausdorff spaces are closed.
- (b) If F is closed and K compact in a Hausdorff space, then $F \cap K$ is compact. *From (a) and 2.4.*

(2.6) If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if they have empty intersection, then so does some finite subcollection of $\{K_\alpha\}$. *Let $V_\alpha := K_\alpha^c$. If $k \in K_1$ then $k \notin K_i$ for some i by assumption, so $k \in V_i^c$. Hence $\{V_\alpha\}$ covers K_1. Now use compactness:*

$$K_1 \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} \implies K_1 \cap (V_{\alpha_1} \cup \ldots \cup V_{\alpha_n})^c = K_1 \cap K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} = \varnothing.$$

(2.7) If X is a locally compact Hausdorff space, $U \subset X$ open, and if $K \subset U$ compact, then there exists an open, precompact V such that $K \subset V \subset \overline{V} \subset U$. *Use local compactness of X and compactness of K to cover K using open sets whose union G is open and precompact. Then for $p \in U^c$, use 2.5 to get $W_p \supset K$ such that $p \notin W_p$. Consider $\{U^c \cap \overline{G} \cap \overline{W}_p\}$, the collection of which has empty intersection. Use 2.6 to derive a finite subcollection and let*

$$V := G \cap W_{p_1} \cap \ldots \cap W_{p_n}.$$

(2.8) A real function f on a topological space is **lower semicontinuous** if $\{x : f(x) > \alpha\}$ is open for every real α; similarly, it is **upper semicontinuous** if $\{x : f(x) < \alpha\}$ is open for every real α. Note that

- (a) Characteristic functions of open sets are lower semicontinuous,
- (b) Characteristic functions of closed sets are upper semicontinuous, and

(c) The supremum of lower semicontinuous functions is lower continuous; the infimum of upper semicontinuous functions is upper semicontinuous.

(2.9) The **support** of a function f on a topological space X is the closure of the set $\{x : f(x) \neq 0\}$. Given a topological space X, the collection of all continuous functions with complex support is denoted by $C_c(X)$.

(2.10) The continuous image of a compact set is compact.

(2.11) Notation (think of characteristic functions):

(a) We write $K \prec f$ if $K \subset X$ is compact, $f \in C_c(X)$, $0 \leqslant f(x) \leqslant 1$ for $x \in X$, and $f(x) = 1$ for all $x \in K$.

(b) We write $f \prec V$ if V is open, $f \in C_c(X)$, $0 \leqslant f \leqslant 1$, and that the support of f is in V.

(2.12) **Urysohn's Lemma**. If X is a locally compact Hausdorff space, $V \subset X$ open, and $K \subset V$ compact, then there exists $f \in C_c(X)$ with $K \prec f \prec V$. (Think $\chi_K \leqslant f \leqslant \chi_V$.)

Let $\{r_n\}$ enumerate all rationals in $[0,1]$ with $r_1 = 0, r_2 = 1$. Use 2.7 to find open sets V_0, V_1 with $K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset V$. Do this inductively for later terms with $\overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i}$ where, among $r_1, ..., r_n$, r_i is the largest rational smaller than r_{n+1} and r_j the smallest rational larger than r_n. Eventually we obtain a countable collection $\{V_r\}$ with $s > r$ implying $\overline{V}_s \subset V_r$. Let

$$f_r(x) = \begin{cases} r & x \in V_r \\ 0 & \text{otherwise} \end{cases} \qquad g_s(x) = \begin{cases} 1 & x \in \overline{V}_s \\ s & \text{otherwise} \end{cases}$$

and define $f := \sup f_r, g = \inf g_s$. It suffices to show $f = g$, and this can be done using the inclusion $\overline{V}_s \subset V_r$.

(2.13) If X is a locally compact Hausdorff space, $V_1, ..., V_n \subset X$ open, K compact, and $K \subset V_1 \cup ... \cup V_n$, then there exists $h_i \prec V_i$ (for $i \in [1, n]$) such that

$$h_1(x) + ... + h_n(x) = 1 \quad \text{for } x \in K.$$

By 2.7, for each $x \in K$ there exists some precompact neighborhood W_x satisfying $\overline{W}_x \subset V_i$ for some i. Use K's compactness to cover it by $W_{x_1} \cup ... \cup W_{x_m}$. Let H_i be the union of W_{x_i}'s whose closures lie in V_i for $i \in [1, n]$. Note that H_i is compact, V_i open, and clearly $H_i \subset V_i$. Use Urysohn's lemma and define g_i such that $H_i \prec g_i \prec V_i$, and define

$$h_1 = g_1 \qquad h_n = (1 - g_1)...(1 - g_{n-1})g_n.$$

Then $h_i \prec V_i$ and $h_1 + ... + h_n = 1 - (1 - g_1)...(1 - g_n)$ by induction. For $x \in K$, it is contained in some H_i so $g_i(x) = 1$, and thus the entire expression evaluates to 1 for all $x \in K$.

(2.14) **Riesz(-Markov-Kakutani) Representation Theorem**. Let X be a locally compact Hausdorff space and let Λ be a positive linear functional on $C_c(X)$ ($\Lambda f \geqslant 0$ whenever $f \geqslant 0$). Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique measure μ on $\mathfrak{M}$ which *represents* Λ in the sense that

(a) $$\Lambda f = \int_X f \, \mathrm{d}\mu \quad \text{for every } f \in C_c(X).$$

Furthermore,

(b) $\mu(K) < \infty$ for every compact $K \subset X$.

(c) For every $E \subset \mathfrak{M}$ we have $\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}$.

(d) $\mu(E) = \sup\{\mu(K) : K \subset E. K \text{ compact}\}$ holds for every open set E and every $E \subset \mathfrak{M}$ with $\mu(E) < \infty$.

(e) If $E \subset \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \subset \mathfrak{M}$.

Proof sketch. We first construct μ and $\mathfrak{M}$ and then show they meet the requirements.

(Construction 1) For open sets $V \subset X$, define $\mu(V) := \sup\{\Lambda f : f \prec V\}$. It follows that if $V_1 \subset V_2$ then $\mu(V_1) \leqslant \mu(V_2)$. In particular this gives an equivalent definition using infimum.

(Construction 2) Generalize μ to all $E \subset X$ using the second definition, i.e.,

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ E \text{ open}\}.$$

(Construction 3) Define $\mathfrak{M}_F$ to be the collection of $E \subset X$ satisfying

$$\mu(E) < \infty \qquad \text{and} \qquad \mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

(Construction 4) Define $\mathfrak{M}$ to be the collection of $E \subset X$ such that $E \cap K \in \mathfrak{M}_F$ for every compact K.

Sketch of main proof:

(Step 1) Show μ is countably subadditive.

(Step 2) Show that if K is compact, then $K \in \mathfrak{M}_F$ and $\mu(K) = \inf\{\Lambda f : K \prec f\}$.

(Step 3) Every open set V with $\mu(V) < \infty$ is an element of $\mathfrak{M}_F$.

(Step 4) μ is countably additive for disjoint members of $\mathfrak{M}_F$ and if the union has finite measure then it is also in $\mathfrak{M}_F$.

(Step 5) If $E \subset \mathfrak{M}_F$ and $\epsilon > 0$ then there exist compact K and open V such that

$$K \subset E \subset V \qquad \text{and} \qquad \mu(V - K) < \epsilon.$$

(Step 6) $\mathfrak{M}_F$ is closed under subtraction, union, and intersection.

(Step 7) $\mathfrak{M}$ is a σ-algebra in X which contains all Borel sets.

(Step 8) $\mathfrak{M}_F = \{E \in \mathfrak{M} : \mu(E) < \infty\}$.

(Step 9) μ is a measure on $\mathfrak{M}$.

(Step 10) For $f \in C_c(x)$, $\Lambda f = \int_X f \, d\mu$, as claimed in (a).

(2.15) A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a **Borel measure** on X. If μ is positive and a Borel set E satisfies 2.14(c) (resp. (d)), then this set is **outer regular** (resp. **inner regular**). μ is **regular** if every Borel set is both outer and inner regular.

(2.16) A set E in a topological space is **σ-compact** if it is a countable union of compact sets.

A set E in (X, μ) has **σ-finite measure** if E is the countable union of sets E_i with $\mu(E_i) < \infty$.

(2.17) If X is a locally compact, σ-compact Haudorff space, then μ and $\mathfrak{M}$ from 2.14 enjoy the following properties:

(a) If $E \in \mathfrak{M}$ and $\epsilon > 0$ then there exist a closed set F and an open set V with $F \subset E \subset V$, $\mu(V - F) < \epsilon$.

(b) μ is a regular Borel measure on X.

(c) If $E \subset \mathfrak{M}$, then there exist $A \in F_\sigma, B \in G_\delta$ with $A \subset E \subset B$ and $\mu(B - A) = 0$ (notation: 1.11).

Proof sketch: write $X = K_1 \cup K_2 \cup \ldots$. *If* $E \in \mathfrak{M}$ *then* $\mu(K_n \cap E) < \infty$, *and for* $\epsilon > 0$ *there exist* $V_n \supset (K_n \cap E)$ *with* $\mu(V_n - (K_n \cap E)) < \epsilon/2^{n+1}$. *Write* $V = \bigcup_{n=1}^{\infty} V_n$ *and so* $V - E \subset \bigcup_{n=1}^{\infty}(V_n - (K_n \cap E))$. *Then* $\mu(V - E) < \epsilon/2$. *Repeat and obtain* $\mu(W - E^c) < \epsilon/2$ *for* W *the union of some open sets* W_n *depending on* E_n^c. *Take* $F = W^c$.

For (b), show borel sets are inner regular: every closed $F \subset X$ *is* σ*-compact because* $F = \bigcup_{n=1}^{\infty}(F \cap K_n)$. *(Out regularity is already given by 2.14.)*

For (c), apply (a) with $\epsilon = 1/n$ *and let* $n \to \infty$. *Take* $A = \bigcup_{n=1}^{\infty} F_n$ *and* $B = \bigcap_{n=1}^{\infty} V_n$.

(2.18) If X is a locally compact Hausdorff space in which every open set is σ-compact, and if λ is any positive Borel measure on X with $\mu(K) < \infty$ for compact K, then λ is regular.

Let $\Lambda f := \int_X f \,\mathrm{d}\lambda$ *for* $f \in C_c(X)$. *By 2.17 there exists* μ *with* $\int_X f \,\mathrm{d}\lambda = \int_X f \,\mathrm{d}\mu$ *for all* $f \in C_c(X)$. *Let* $V \subset X$ *be open. Write* $V = \bigcup_{i=1}^{\infty} K_i$. *By Urysohn's lemma, choose* f_i *with* $K_i \prec f_i \prec V$, *and define* $g_n := \max\{f_1, \ldots, f_n\}$. *Then* $g_n \in C_c(X)$ *and* $g_n \to \chi_V$. *Use MCT to get*

$$\lambda(V) = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = \mu(V).$$

Let E *be a Borel set and use 2.17(a) to complete the proof. Construct* $F \subset E \subset V$ *(*F *closed,* V *open) with* $\mu(V - F) < \epsilon$. *Then*

$$\begin{cases} \lambda(E) \leqslant \lambda(V) = \mu(V) \leqslant \mu(E) + \epsilon \\ \mu(E) \leqslant \mu(V) = \lambda(V) \leqslant \lambda(E) + \epsilon \end{cases} \implies \lambda(E) = \mu(E),$$

where the second inequality $\lambda(V) \leqslant \lambda(E) + \epsilon$ *is becuase* $V - F$ *is open and thus* $\mu(V - F) = \lambda(V - F) < \epsilon$.

(2.19) Let P_n be the set of all $x \in \mathbb{R}^k$ with coordinates being multiples of 2^{-n}. Let Ω_n be the collection of all 2^{-n} boxes with corners at points of P_n. Then every nonempty open set in $\mathbb{R}^k$ is a countable union of disjoint boxes from $\Omega_1 \cap \Omega_2 \cap \ldots$. *Write* V *as the union of balls centered at each* $x \in V$. *Start with boxes in* Ω_1 *and remove all balls with centers in these biggest boxes and iterate the process.*

(2.20) **Lebesgue measure**. There exists a positive complete measure m defined on a σ-algebra $\mathfrak{M}$ in $\mathbb{R}^k$ satisfying

(a) $m(W) = \mathrm{vol}(W)$ for every k-cell W.

(b) $\mathfrak{M}$ contains all Borel sets in $\mathbb{R}^k$. In particular $E \in \mathfrak{M}$ if and only if there are $A \in F_\sigma, B \in G_\delta, A \subset E \subset B$, and $m(B - A) = 0$. Also m is regular.

(c) m is translation-invariant, i.e., $m(E + x) = m(E)$.

(d) If μ is any positive translation-invariant Borel measure on $\mathbb{R}^k$ such that $\mu(K) < \infty$ for every compact K, then $\mu(E) = cm(E)$ for some c and all Borel sets E.

(e) To every linear transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ corresponds $\Delta(T) \in \mathbb{R}$ with

$$m(T(E)) = \Delta(T)m(E)$$

for every $e \in \mathfrak{M}$. If T is a rotation then $\Delta(T) = 1$.

The members of $\mathfrak{M}$ are called the **Lebesgue measurable sets** in $\mathbb{R}^k$ and m is called the **Lebesgue measure**.

Proof sketch: first define $\Lambda_n f := 2^{-nk} \sum_{x \in P_n} f(x)$ *for* $n \in \mathbb{N}$. *If* $f \in C_c(\mathbb{R}^k)$, f *is real, and* $W \supset \operatorname{supp}(f)$ *an open* k*-cell, then the uniform continuity of* f *(continuous with compact domain) implies we can bound* f *by locally on (each box in)* Ω_N *by two (box-piecewise) constant functions differing less than* ϵ*, i.e., (1)* g, f *constant on each box* Ω_N*, (2)* $g \leqslant f \leqslant h$*, and (3)* $h - g < \epsilon$*. Using (1) and (2),*

$$\Lambda_N g = \Lambda_n g \leqslant \Lambda_n f \leqslant \Lambda_n h = \Lambda_N h.$$

Letting $\epsilon \to 0$ *we have* $\Lambda_N h - \Lambda_N g < \epsilon \cdot \operatorname{vol}(W) \to 0$ *so* $\Lambda f := \lim_{n\to\infty} \Lambda_n f$ *exists for* $f \in C_c(\mathbb{R}^k)$.

Now define m *and* $\mathfrak{M}$ *using 2.14. Since* $\mathbb{R}^k$ *is* σ*-compact, 2.17 implies (b).*

Now for (a): let W *be an open cell. Use 2.19 and define* E_r *to be the union of those boxes belonging to* Ω_r*. Choose* f_r *with* $\overline{E}_r \prec f_r \prec W$ *and define* $g_r = \max\{f_1, \ldots, f_r\}$*. Then* $\operatorname{vol}(E_r) \leqslant \Lambda f_r \leqslant \Lambda g_r \leqslant \operatorname{vol}(W)$ *and*

$$\Lambda g_r = \int_{E_r} g_r \, \mathrm{d}m \to m(W) \quad \textit{as } r \to \infty \textit{ by MCT since } g_r \to \chi_W.$$

Thus (a) holds for open cells W*. Now use* $[a,b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n)$ *to get (a).*

To prove (c), define $\lambda(E) := m(E + x)$*. Then* λ *is a measure,* $\lambda(E) = m(E)$ *by (a), and thus* $m(E + x) = m(E)$ *for all Borel sets. Then use (b) for arbitrary sets.*

For (d), let μ *be such a measure and let* $c := \mu(Q_0)$ *where* Q_0 *is the unit box. Then for* 2^{-n}*-boxes,*

$$2^{nk}\mu(Q) = \mu(Q_0) = cm(Q_0) = c2^{nk}m(Q).$$

Use 2.19 to generalize this to all open sets and thus all Borel sets.

For (e), if $\dim(\operatorname{Range}(T)) < k$ *then* $\Delta(T) = 0$*. Otherwise define a new measure* $\mu(E) := m(T(E))$*:*

$$\mu(E + x) = m(T(E + x)) = m(T(E) + Tx) = m(T(E)) = \mu(E),$$

and $\Delta(T)$ *is simply* $m(T(E))/m(E)$*. If* T *is rotational then letting it act on the unit ball gives coefficient* 1.

(2.21) A Lebesgue measurable set is not a Borel set in general. "Most" subsets of $\mathbb{R}^k$ are not Lebesgue measurable. *For the first claim, consider subsets of a Cantor set. The cardinality of* $\mathscr{B}$*, the collection of Borel sets generated by a countable base, is* $\mathfrak{c}$*, whereas the collection of subsets of the Cantor set is* $2^{\mathfrak{c}}$*, but each of them is indeed measurable, as guaranteed by the completeness of* m.

(2.22) If $A \subset \mathbb{R}$ and every subset of A is Lebesgue measurable then $m(A) = 0$.

(2.23) 2.20's $\Delta(T)$ is equal to $|\det(T)|$, where T is represented in its matrix form, i.e.,

$$Te_j = \sum_{i=1}^{k} \alpha_{i,j} e_i \implies T_{i,j} = \alpha_{i,j}.$$

It suffices to prove the claim for permutation, scalar multiplication of a row, and addition of two rows.

(2.24) **Lusin's Theorem**. If f is complex measurable on X, $\mu(A) < \infty$, $f(x) = 0$ outside A, then for all $\epsilon > 0$ there exists $g \in C_c(X)$ with

$$\mu(\{x : f(x) \neq g(x)\} < \epsilon.$$

In particular, there exists such g satisfying $\sup|g(x)| \leqslant \sup|f(x)|$.

Corollary. Assuming above and that $|f| \leqslant 1$. Then there exists sequence $\{g_n\}$ with $g_n \in C_c(X), |g_n| \leqslant 1$, and

$$f(x) = \lim_{n\to\infty} g_n(x) \quad \text{a.e.}$$

For the corollary, for each n there exists g_n such that $\mu(E_n) < 2^{-n}$ where E_n is the set on which f and g_n differs. Then $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, and 1.41 says that almost all $x \in X$ are in finitely many E_n's. For sufficiently large n, f and g_n agree at these points, hence "a.e."

(2.25) **Vitali-Carathéodory Theorem**. Suppose $f \in L^1(\mu)$ and f is real valued. Then, given ϵ, there exist functions u, v such that $u \leqslant f \leqslant v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and

$$\int_X (v - u)\, \mathrm{d}\mu < \epsilon.$$