Rudin RCA Chapter 3 Exercises

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July 13, 2021

Problem 1

Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) [if it is finite] and that the pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of convex functions?

Proof. (1) For (pointwise) supremum: let $\{f_n\}$ be a sequence of convex functions on (a, b) and let $x < y$ be any two points in (a, b). Also let $\lambda \in [0, 1]$. Then, for any n, we have

$$(1-\lambda)f(x) + \lambda f(y) \geqslant (1-\lambda)f_n(x) + \lambda f_n(y) \geqslant f_n((1-\lambda)x + \lambda y).$$

(The first $\leqslant$ is because f is the pointwise supremum so $f \geqslant f_n$ and the second by convexity of f_n.) Taking the supremum of the RHS once again, we obtain

$$(1-\lambda)f(x) + \lambda f(y) \geqslant \sup_n f_n((1-\lambda)x + \lambda y) = f((1-\lambda)x + \lambda y),$$

from which the convexity of f follows.

(2) For pointwise limit: let $\{f_n\}$, x, y, and λ be defined as above and let f be the pointwise limit of $\{f_n\}$. By convexity of the f_n's,

$$(1-\lambda)f_n(x) + \lambda f_n(y) \geqslant f_n((1-\lambda)x + \lambda y).$$

Since $f_n(x) \to f(x)$, $f_n(y) \to f(y)$, and $f_n((1-\lambda)x + \lambda y) \to f((1-\lambda)x + \lambda y)$ as $n \to \infty$, we obtain

$$(1-\lambda)f(x) + \lambda f(y) \geqslant f((1-\lambda)x + \lambda y),$$

so f is convex.

(3) Upper and lower limits: for limsup, simply notice that

$$f(x) = \limsup_{n\to\infty} f_n(x) = \lim_{k\to\infty}\left(\sup_{n\geqslant k} f_n(x)\right)$$

and the convexity of $\limsup f_n$ follows from using (1) and then (2). However, $\liminf f_n$ need not to be convex: if we let $f_n(x) := (-1)^n x$ on $(-1, 1)$ (or any interval containing the origin) then clearly $\liminf f_n$ is $-|x|$ which is strictly concave. □

Problem 2

If φ is convex on (a, b) and if ψ is convex and nondecreasing on the range of φ, prove that $\psi \circ \varphi$ is convex on (a, b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ but not vice versa.

Proof. (1) Let $a < x < y < b$ for some x, y and let $\lambda \in [0, 1]$. For simplicity denote $z := (1-\lambda)x + \lambda y$. Since φ is convex,

$$\varphi(z) \leqslant (1-\lambda)\varphi(x) + \lambda\varphi(y).$$

Therefore, by the monotonicity of ψ (first $\leqslant$ below) and convexity of ψ (second),

$$\psi(\varphi(z)) \leqslant \psi((1-\lambda)\varphi(x) + \lambda\varphi(y)) \leqslant (1-\lambda)\psi(\varphi(x)) + \lambda\psi(\varphi(y))$$

which completes the proof.

(2) Notice that $\varphi = \exp(\log \varphi)$, and clearly $\exp$ is a convex function. The convexity of φ therefore follows from (1). The converse is obviously false — $\mathrm{id} : x \mapsto x$ is convex but $\log(\mathrm{id}(x)) = \log(x)$ is not. □

Problem 3

Assume that φ is a continuous real function on (a, b) such that

$$\varphi\left(\frac{x+y}{2}\right) \leqslant \frac{\varphi(x)}{2} + \frac{\varphi(y)}{2}$$

for all x, y in (a, b). Prove that φ is convex. (The claim does not follow if φ is not continuous.)

Proof. Notice that the inequality holds not only for $(x+y)/2$ but also $(1-q)x+qy$ for any dyadic number $q \in [0, 1]$. For example, we need to iterate the inequality twice to obtain the case for $q = 3/4$:

$$\varphi\left(\frac{x}{4} + \frac{3y}{4}\right) \leqslant \frac{1}{2} \cdot \varphi\left(\frac{x+y}{2}\right) + \frac{\varphi(y)}{2} \leqslant \frac{\varphi(x)}{4} + \frac{\varphi(y)}{4} + \frac{\varphi(y)}{2} = \frac{\varphi(x)}{4} + \frac{3\varphi(y)}{4}.$$

In general, if q is dyadic, then

$$\varphi((1-q)x + dy) \leqslant (1-q)\varphi(x) + q\varphi(y). \qquad (\Delta)$$

Let $\epsilon > 0$ and $\lambda \in [0, 1]$ be given. Since the dyadic rationals are dense in $\mathbb{R}$, there exists a dyadic sequence $\{q_n\}$ that converges to λ. Since φ is continuous, it preserves sequential limits. Thus,

$$\begin{aligned} \varphi((1-\lambda)x + \lambda y) &= \varphi((1 - \lim_{n\to\infty} q_n)x + \lim_{n\to\infty} q_n y) \\ &= \lim_{n\to\infty} \varphi((1-q_n)x + q_n y) \\ (\Delta) &\leqslant \lim_{n\to\infty} (1-q_n)\varphi(x) + q_n\varphi(y) \\ &= (1-\lambda)\varphi(x) + \lambda\varphi(y). \end{aligned}$$

□

Problem 4

Suppose f is a complex measurable function on X, μ a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty).$$

Let $E := \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leqslant \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $< \infty$. Prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

Proof. (a) Since $r < p < s$, we can express p as a convex combination of r, s; namely, there exists $\lambda \in (0,1)$ satisfying $p = (1-\lambda)r + \lambda s$. Hölder's inequality on the conjugate exponent $1/(1-\lambda)$ and $1/\lambda$ gives

$$\begin{aligned}
\varphi(p) &= \int_X |f|^p \, d\mu = \int_X |f|^{(1-\lambda)r+\lambda s} \, d\mu \\
&= \int_X |f|^{(1-\lambda)r} |f|^{\lambda s} \, d\mu \\
[\text{Hölder's}] &\leqslant \left\{ \int_X \left(|f|^{(1-\lambda)r}\right)^{1/(1-\lambda)} \, d\mu \right\}^{1-\lambda} \left\{ \int_X \left(|f|^{\lambda s}\right)^{1/\lambda} \, d\mu \right\}^{\lambda} \\
&= \left\{ \int_X |f|^r \, d\mu \right\}^{1-\lambda} \left\{ \int_X (|f|^s) \, d\mu \right\}^{\lambda} = \varphi(r)^{1-\lambda} \varphi(s)^{\lambda} < \infty.
\end{aligned}$$

(b) If E has empty interior then the claim holds trivially. Otherwise, E is connected and its interior can only be of form (a, b) for some $a < b$. Let $x < y$ be two points in (a, b) and let $\lambda \in [0, 1]$. Since $\|f\|_\infty > 0$ by assumption, we claim that $\varphi > 0$ as well. Indeed, if

$$\varphi(p) = \int_X |f|^p \, d\mu = 0,$$

then $|f|^p = 0 = |f|$ almost everywhere, and its essential supremum $= 0$, contradicting $\|f\|_\infty > 0$. From (a),

$$\varphi((1-\lambda)x + \lambda y) \leqslant \varphi(x)^{1-\lambda} \varphi(y)^{\lambda},$$

and since $\log$ is monotone we can take logarithm of both sides while preserving the $\leqslant$:

$$\begin{aligned}
\log \varphi((1-\lambda)x + \lambda y) &\leqslant \log \varphi(x^{1-\lambda}) + \log \varphi(y^{\lambda}) \\
&= (1-\lambda) \log \varphi(x) + \lambda \log \varphi(y)
\end{aligned}$$

so $\log \varphi$ is convex.

To show that φ is continuous on E, first notice that it is continuous on (a,b), the interior of E: the convexity of $\log\varphi$ on (a,b) implies that of φ by Problem 2, and this further implies continuity of φ on (a,b). It remains to show that φ is continuous on the endpoints of E [if it has any]. WLOG let us assume E is at least left-closed [i.e., it is of form $[a,b]$ or $[a,b)$]. We pick $\epsilon > 0$ such that $[a, a+\epsilon) \subset E$. Then, if $p \in [a, a+\epsilon)$, we have

$$|f(x)|^p \leqslant \begin{cases} |f(x)|^a & \text{if } |f(x)| \leqslant 1 \\ |f(x)|^{a+\epsilon} & \text{if } |f(x)| > 1, \end{cases} \implies |f(x)|^p \leqslant |f(x)|^a + |f(x)|^{a+\epsilon},$$

so in $L^1(\mu)$, $|f|^p$ is bounded absolutely by $|f|^a + |f|^{a+\epsilon}$ [which is also L^1]. Clearly as $p \to a$, $|f(x)|^p \to |f(x)|^a$, so by Lebesgue's DCT, we have

$$\lim_{p\to a} \varphi(p) = \lim_{p\to a} \int_X |f|^p \, \mathrm{d}\mu = \int_X |f|^a \, \mathrm{d}\mu = \varphi(a),$$

and so φ is indeed continuous on the boundary of E, if it has any.

(c) (1) An example where $E = (a,b)$: define $f : (0,\infty) \to (0,\infty)$ by

$$f(x) := \begin{cases} x^{-1/b} & x \in (0,1) \\ x^{-1/a} & x \geqslant 1. \end{cases}$$

It follows that

$$\int_0^\infty |f(x)|^p \, \mathrm{d}x = \int_0^1 |f(x)|^p \, \mathrm{d}x + \int_1^\infty |f(x)|^p \, \mathrm{d}x = \int_0^1 1/x^m \, \mathrm{d}x + \int_1^\infty 1/x^n \, \mathrm{d}x$$

which converges if and only if $m = p/b < 1$ and $n = p/a > 1$, that is, $p \in (a,b)$.

(2) An example where $E = [1,\infty)$: consider $f(x) := 1/(x \log^2(x))$ on $[3,\infty)$. If $p < 1$, then the series

$$\sum_{n=4}^\infty a_n := \sum_{n=4}^\infty \frac{1}{n^p \log^{2p}(n)}$$

diverges by the Cauchy condensation test because $\sum_{k=4}^{n} 2^n a_{2^n}$ diverges:

$$\lim_{n\to\infty} \frac{2^{n+1} a_{2^{n+1}}}{2^n a_{2^n}} = \lim_{n\to\infty} 2 \cdot \frac{2^{np}(n \log 2)^{2p}}{2^{(n+1)p}((n+1)\log 2)^{2p}}$$
$$\sim \lim_{n\to\infty} 2 \cdot 2^{-p} > 1.$$

On the other hand, if $p = 1$, then

$$\int_3^\infty \frac{1}{x \log^2(x)} \, \mathrm{d}x = \frac{1}{\log(x)}\Big]_{x=3}^\infty = \frac{1}{\log 3} < \infty,$$

and if $1 < p < \infty$, then the integral is dominated by that of $p = 1$, so $E = [1,\infty)$. (*This is the only reason why I chose the domain* $[3,\infty)$*:* $1/(3\log^2(3)) < 1$ *and the function is strictly decreasing hereafter.*)

(3) A tentative solution to E taking the form $[a,b]$: since $[a,b] = \bigcap_{n=1}^\infty (a - 1/n, b + 1/n)$, we define

$$f_n(x) := \begin{cases} x^{-1/(b+1/n)} & 0 < x < 1 \\ x^{-1/(a-1/n)} & x \geqslant 1 \end{cases} \qquad \text{and} \qquad \tilde{f}(x) := \sum_{n=1}^\infty 2^{-n} f_n(x).$$

(We still let f be defined as in (a).) Since on $(0,1)$, $x^{-1/k}$ decreases as k increases, $\tilde{f}\big|_{(0,1)}$ is bounded by

$$\sum_{n=1}^{\infty} 2^{-n} x^{-1/b} = x^{-1/b},$$

and similarly $\tilde{f}\big|_{[1,\infty)}$ is bounded by $x^{-1/a}$. Therefore $\tilde{f}$ is well-defined and bounded by f. If $p \in (a,b)$, then

$$\|\tilde{f}\|_p^p = \int_0^\infty \tilde{f}(x)^p \,\mathrm{d}x \leqslant \int_0^\infty f(x)^p \,\mathrm{d}x = \|f\|_p^p < \infty \qquad \text{by (a).}$$

If $p \notin [a,b]$, WLOG we can assume $p < a$. Then there exists sufficiently large n such that $p < a - 1/n$, and so

$$\|\tilde{f}\|_p^p = \int_0^\infty \tilde{f}(x)^p \,\mathrm{d}x \geqslant \int_0^1 \tilde{f}(x)^p \,\mathrm{d}x \geqslant 2^{-n} \int_0^1 f_n(x) \,\mathrm{d}x = 2^{-n} \int_0^1 1/x^{p/(a-1/n)} \,\mathrm{d}x.$$

Since $p > a - 1/n$, the last integral diverges, and thus $\tilde{f} \notin L^p$.

Now we discuss the case where $p \in \{a,b\}$. WLOG let $p = a$. Notice that

$$\int_1^\infty |x^{-1/(a-1/n)}|^p \,\mathrm{d}x = \int_1^\infty x^{-a/(a-1/n)} \,\mathrm{d}x = (1-an)x^{1/(1-an)}\Big]_{x=1}^{\infty} = an - 1.$$

Splitting $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$, we obtain

$$\sum_{n=1}^{\infty} 2^{-n} \|f_n\|_p^p \leqslant \sum_{n=1}^{\infty} 2^{-n} \|f_n\big|_{(0,1)}\|_p^p + \sum_{n=1}^{\infty} 2^{-n} \|f_n\big|_{[1,\infty)}\|_p^p. \tag{i}$$

The first term is finite, as shown in (a) as $p/b = a/b < 1$, and the second is $\sum_{n=1}^{\infty} \frac{an-1}{2^n} = 2a - 1$, also finite. Finally, to prove the claim, it remains to notice that

$$\|f\|_p^p = \Big\| \sum_{n=1}^{\infty} 2^{-n} f_n \Big\|_p^p \leqslant \sum_{n=‘}^{\infty} 2^{-n} \|f_n\|_p^p. \tag{ii}$$

If we define the partial sums $S_k := \Big| \sum_{n=1}^{k} g_n \Big|^p$ where $g_n := 2^{-n} f_n$, then Fatou's lemma gives

$$\Big\| \sum_{n=1}^{\infty} g_n \Big\|_p^p = \int_0^\infty \lim_{n\to\infty} S_n \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_0^\infty S_n \,\mathrm{d}x = \liminf_{n\to\infty} \Big\| \sum_{k=1}^{n} g_k \Big\|_p^p,$$

so $\Big\| \sum_{n=1}^{\infty} g_n \Big\|_p \leqslant \liminf_{n\to\infty} \Big\| \sum_{k=1}^{n} g_k \Big\|_p$. Since the RHS consists of finite sums, Minkowski's inequality gives

$$\|f\|_p = \Big\| \sum_{n=1}^{\infty} g_n \Big\|_p \leqslant \liminf_{n\to\infty} \sum_{k=1}^{n} \|g_k\|_p = \sum_{n=1}^{\infty} \|g_n\|_p.$$

This proofs (ii). Combining (i) and (ii), we see that $\|f\|_p$ is finite, and this completes the proof.

(4) An example where E is a singleton: the example comes from Oźański's HW: in $\mathbb{R}^n$,

$$f(x) := |x|^{-n/p}(1 + \log^2|x|)^{-1/p}$$

corresponds to $E = \{p\}$.

(5) E can also be empty: $\int_{\mathbb{R}} |f|^p \,\mathrm{d}x$ where $f \equiv 1$ is never finite.

(6) Finally, E can be any connected subset of $(0, \infty)$, since any such set must be of form $[a, b], (a, b), \{p\}, [a, b)$, or $(a, b]$. We've shown the first three, and the last two can be obtained by

$$[a, b) = [a, c] \cap (d, b) \text{ and } (a, b] = (a, c) \cap [d, b]$$

where $d < a < b < c$. Let f_1 be the function corresponding to $[a, c]$ (resp. (a, c)) and let f_2 correspond to (d, b) (resp. $[d, b]$). Then $f_1 + f_2$ is L^p if and only if p is in both sets, namely $p \in [a, b)$ (resp. $(a, b]$).

(d) Defining λ as in (a), we have

$$\|f\|_p^p \leqslant \|f\|_r^{(1-\lambda)r} \|f\|_s^{\lambda s} \leqslant \begin{cases} \|f\|_r^{(1-\lambda)r+\lambda s} = \|f\|_r^p & \text{if } \|f\|_r^r \geqslant \|f\|_s^s \\ \|f\|_s^{(1-\lambda)r+\lambda s} = \|f\|_s^p & \text{if } \|f\|_r^r < \|f\|_s^s. \end{cases}$$

If $f \in L^r(\mu) \cap L^s(\mu)$ then both $\|f\|_r$ and $\|f\|_s$ are finite and thus so is $\|f\|_p$.

(e) It suffices to show that (i) $\liminf_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty$ and (ii) $\limsup_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty$. Of course, the claim is trivial if $\|f\|_\infty = \infty$, since then $\|f\|_p = \infty$ for all p. Now let us assume $\|f\|_\infty < \infty$.

To show (i), we pick an arbitrary $k \in (0, \|f\|_\infty)$ and consider the set

$$S := \{x \in X : |f(x)| \geqslant k\}.$$

This set must have finite measure (or else $\|f\|_r \geqslant \mu(S)k^r$, contradicting the assumption that $\|f\|_r < \infty$), so

$$\|f\|_p^p = \int_X |f|^p \, d\mu \geqslant \int_S k^p \, d\mu = \mu(S)k^p,$$

so $\|f\|_p \geqslant k\mu(S)^{1/p}$. Letting $p \to \infty$ we obtain $\liminf_{p\to\infty} \|f\|_p \geqslant k$. Since k is arbitrary, we recover (i).

For (ii), if $p > r$, define $E := \{x \in X : |f(x)| > \|f\|_\infty\}$, a null set. Then we have

$$\begin{aligned} \|f\|_p^p = \int_X |f|^p \, d\mu &= \int_{X-E} |f|^{p-r}|f|^r \, d\mu \\ &\leqslant \int_{X-E} \|f\|_\infty^{p-r}|f|^r \, d\mu = \|f\|_\infty^{p-r} \int_{X-E} |f|^r \, d\mu = \|f\|_\infty^{p-r}\|f\|_r^r, \end{aligned}$$

so

$$\|f\|_p \leqslant \|f\|_\infty^{(p-r)/p} \|f\|_r^{r/p}.$$

Taking $\limsup_{p\to\infty}$, we obtain $\limsup_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty \cdot 1$, as claimed. □

Problem 5

Assume, in addition to the hypotheses of Problem 4, that $\mu(X) = 1$.

(a) Prove that $\|f\|_r \leqslant \|f\|_s$ if $0 < r < s \leqslant \infty$.

(b) Under what conditions does it happen that $0 < r < s \leqslant \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$. Prove that

$$\lim_{p\to 0} \|f\|_p = \exp\left\{\int_X \log|f| \, d\mu\right\}$$

if $\exp(-\infty) := 0$.

Proof. (a) If $s = \infty$ then the set $S := \{x \in X : |f(x)| > \|f\|_\infty\}$ is of measure 0, so

$$\|f\|_r^r = \int_X |f|^r \, d\mu = \int_{X-S} |f|^r \, d\mu \leqslant \int_{X-S} \|f\|_\infty^r \, d\mu = \|f\|_\infty^r.$$

Otherwise, this is also a one-liner with Hölder's inequality:

$$\|f\|_r^r = \int_X |f|^r \cdot 1 \, d\mu \leqslant \left\{\int_X (|f|^r)^{s/r} \, d\mu\right\}^{r/s} \underbrace{\left\{\int_X 1^{s/(s-r)} \, d\mu\right\}^{1-r/s}}_{=1} = \|f\|_s^r.$$

(b) To attain equality in Hölder's inequality in the previous part, equation (5) in Rudin's Theorem 3.5 must hold almost everywhere (since $\exp$ is strictly convex). Since one of the functions here is merely 1, the other (namely f) must be constant almost everywhere.

(c) The first inclusion follows directly from (a).

For the second claim, we need the opposite inclusion $L^r(\mu) \subset L^s(\mu)$. Borrowing a hint from Folland's Problem 6.5, we claim that

For $r < s$, $L^r(\mu) \subset L^s(\mu)$ if and only if X does not contain sets with arbitrarily small measure (i.e., either a null set or a set with measure $\geqslant \epsilon$).

Proof of claim. For the forward direction, let us assume the contrapositive that X does contain sets with arbitrarily small positive measure. Let E_n be a set with $\mu(E_n) < 2^{-n}$ and WLOG assume $\{E_n\}$ is pairwise disjoint (this can be done by letting $F_1 := E_1$ and $F_n := E_n - (E_1 \cup \ldots \cup E_{n-1})$ if needed). If $s < \infty$, we define

$$f := \sum_{n=1}^\infty \mu(E_n)^{-1/s} \chi_{E_n},$$

we obtain

$$\begin{aligned}\|f\|_r^r = \int_X |f|^r \, d\mu &= \int_E |f|^r \, d\mu \\ &= \sum_{n=1}^\infty \int_{E_n} \mu(E_n)^{-r/s} \, d\mu = \sum_{n=1}^\infty \mu(E_n)^{1-r/s} < \frac{2^{1-r/s}}{1-2^{1-r/s}} < \infty\end{aligned}$$

(since $1 - r/s < 1$). Therefore $f \in L^r(\mu)$.

However, this implies $f \notin L^s(\mu)$, as

$$\begin{aligned}\|f\|_s^s = \int_X |f|^s \, d\mu &= \int_E |f|^s \, d\mu \\ &= \sum_{n=1}^\infty \int_{E_n} \mu(E_n)^{-1} \, d\mu = \sum_{n=1}^\infty 1 = \infty.\end{aligned}$$

This proves $\Rightarrow$. (*If $s = \infty$, a similar argument holds if we define* $f := \sum_{n=1}^{\infty} \mu(E_n)^{-1/(r+1)} \chi_{E_n}$. *Then* $\|f\|_r^r$ *is still finite and* $\|f\|_s^s$ *is still unbounded.*)

For $\Leftarrow$, let $f \in L^r(\mu)$ and consider the sets $E_n := \{x \in X : |f(x)| \geqslant n\}$. Clearly they form a decreasing sequence, and by assumption, either $\mu(E_n) \geqslant \epsilon$ for all n or the tail of this sequence is all 0, i.e., $\mu(E_n) = 0$ for all large n's. If the former is true, then

$$E := \bigcap_{n=1}^{\infty} E_n \text{ satisfies } \mu(E) = \lim_{n\to\infty} \mu(E_n) \geqslant \epsilon$$

(since $\mu(E_1)$ is finite and μ is continuous from above). But then $\|f\|_r^r \geqslant \int_E |f|^r \,\mathrm{d}\mu = \infty$, contradiction. Hence $\mu(E_k) = 0$ for some k (and all later terms too). Therefore $\|f\|_\infty \leqslant k$, and by (a) this implies $\|f\|_s \leqslant k$, so $f \in L^s(\mu)$.

(*If $s = \infty$, suppose some $f \in L^p(\mu)$ but not $L^\infty(\mu)$. Then the corresponding E_n's form a decreasing sequence, but none of them is empty. However,*

$$\|f\|_r^r = \int_X |f|^r \,\mathrm{d}\mu \geqslant \int_{E_n} |f|^r \,\mathrm{d}\mu \geqslant n^r \mu(E_n)$$

so $\mu(E_n) \to 0$ as $n \to \infty$. Contrapositive is proven.) END OF PROOF OF CLAIM

□

Problem 7

For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give example of these situations, and find conditions on μ under which these situations will occur.

Solution. Let $0 < r < s$.

(1) An example where $L^r(\mu) \subset L^s(\mu)$: let X be $\mathbb{N}$ (starting from 1) and let μ be the counting measure. We end up having ℓ^r and ℓ^s, spaces of r^{th}- and s^{th}-power summable sequences. We claim $\ell^r \subset \ell^s$.

Let $x := \{x_n\}_{n\geqslant 1}$ be a ℓ^r sequence, i.e., $\|x\|_r := \left(\sum_{n=1}^{\infty} |x_n|^r\right)^{1/r} < \infty$.

(i) If $\|x\|_r = 1$, it follows that $\|x\|_r^r = \sum_{n=1}^{\infty} |x_n|^r = 1$, so $|x_n|^r \leqslant 1$ — and thus $|x_n| \leqslant 1$ — for all n. Since $r < s$ we have $|x_n|^r \geqslant |x_n|^s$, so $\|x\|_r \geqslant \|x\|_s$, which implies $x \in \ell^s$.

(ii) If $\|x\|_r \neq 1$, we can normalize it by setting $x' := x/\|x\|_r$. Then $\|x'\|_r = 1$ and $x' \in \ell^s$ by (i). Since $\|x\|_r$ is just a scalar, $\|x\|_r \cdot x'$, namely x, is also in ℓ^s.

Therefore $\ell^r \subset \ell^s$.

(2) An example where $L^r(\mu) \supset L^s(\mu)$: see Problem 5(a)/(c).

(3) An example where $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$: consider Problem 4(c). For any $r \neq s$ (assuming WLOG $r < s$), we can construct a function whose corresponding E is $[r-1, r]$ (so $r \notin E$), and such function is L^r but not L^s.

(4) We have characterized measures on which $r < s$ implies $L^r(\mu) \subset L^s(\mu)$ in Problem 5(c); we said it holds if and only if X does not contain sets with arbitrarily small but positive measure. Similarly:

> For $r < s < \infty$, $L^r(\mu) \supset L^s(\mu)$ if and only if X does not contain sets with arbitrarily large (but finite) measure. If $s = \infty$ the "only if" fails.

Proof of claim with $s < \infty$. For $\Rightarrow$, let us consider the contrapositive and suppose that X contains sets with arbitrarily large measure. We define a sequence of sets $\{E_n\}_{n\geqslant 1}$ inductively by

$$E_1 \subset X \qquad \mu(E_{n+1}) > \sum_{k=1}^{n} \mu(E_k) + 1. \tag{Δ}$$

Then we define $\{F_n\}_{n\geqslant 1}$ by

$$F_1 := E_1 \qquad F_{n+1} := E_{n+1} - \bigcup_{k=1}^{n} E_k.$$

Because of (Δ), the F_n's are nonempty and disjoint, and [most importantly] $\mu(F_n) > 1$. We define

$$f := \sum_{n=1}^{\infty} n^{-1/r} \mu(F_n)^{-1/r} \chi_{F_n}.$$

Since

$$\|f\|_r^r = \int_X |f|^r \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{F_n} n^{-r/r} \mu(F_n)^{-r/r} \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

but

$$\|f\|_s^s = \int_X |f|^s \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{F_n} n^{-s/r} \mu(F_n)^{-s/r} \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{\mu(F_n)^{1-s/r}}{n^{s/r}} < \sum_{n=1}^{\infty} \frac{1}{n^{s/r}} < \infty$$

(where the last $<$ is because $s/r > 1$, and the second last because $\mu(F_n) > 1$), we see $L^r(\mu) \not\supset L^s(\mu)$.

For $\Leftarrow$, let $g \in L^s(\mu)$ and $G_n := \{x \in X : 1/(n+1) \leqslant |g(x) < 1/n|\}$. First observe that they are disjoint and for each n

$$\|g\|_s^s = \int_X |g|^s \, \mathrm{d}\mu \geqslant \int_{G_n} |g|^s \, \mathrm{d}\mu \geqslant \int_{G_n} \frac{1}{(n+1)^s} \, \mathrm{d}\mu = \frac{\mu(G_n)}{(n+1)^s}$$

so

$$\mu(G_n) \leqslant \|g\|_s^s (n+1)^s < \infty.$$

Now we define $H_n := \bigcup_{k=1}^{n} G_n$. A finite union of sets of finite measure, clearly $\mu(H_n) < \infty$ for all n. Also notice that $\{H_n\}$ forms a nested increasing sequence. If we define $G := \bigcup_{n=1}^{\infty} G_n$, then

$$\sum_{n=1}^{\infty} \mu(G_n) = \mu(G) = \lim_{n\to\infty} \mu(H_n) \leqslant \sup_{n\geqslant 1} \mu(H_n) < \infty.$$

(The first $=$ is because μ is countably additive (and G_n's are disjoint), the second because of Rudin's Theorem 1.19(d), and the $<$ because of our original assumption.) Also, on $X - G$, $|f| \geqslant 1$, so $|g|^s \geqslant |g|^r$. Then,

$$\begin{aligned} \|g\|_r^r = \int_X |g|^r \, \mathrm{d}\mu &= \int_G |g|^r \, \mathrm{d}\mu + \int_{X-G} |g|^r \, \mathrm{d}\mu \\ &\leqslant \sum_{n=1}^{\infty} \frac{\mu(G_n)}{n^r} + \int_{X-G} |g|^s \, \mathrm{d}\mu < \sum_{n=1}^{\infty} \mu(G_n) + \|f\|_s^s < \infty, \end{aligned}$$

so $f \in L^r(\mu)$, and we are done. END OF PROOF FOR CASE $s < \infty$

Claim with $s = \infty$. The $\Rightarrow$ direction is still valid. Take the contrapositive. If X contains sets with arbitrarily large measure, we can pick a sequence $\{E_n\}$ with $\mu(E_n) \to \infty$. The function $f := \sum_{n=1}^{\infty} \chi_{E_n}$ is obviously L^∞ but not L^r for any finite r. The $\Leftarrow$ fails: let $X := \{x\}$, $\mathfrak{M} := \{\varnothing, X\}$, $\mu(\varnothing) = 0$, and $\mu(X) = \infty$. Then any constant function on X is L^∞ but not L^p, so $L^\infty(\mu) \not\supset L^r(\mu)$ can happen even if X does not contain sets with arbitrarily large (finite) measure. End of Proof for Case $s = \infty$

For the last question, we simply need to combine our previous characterizations — if $L^r(\mu) \not\subset L^s(\mu)$ then there are arbitrarily small subsets; similarly, there are also arbitrarily large subsets, assuming $s < \infty$. □

Problem 8

If g is a positive function on $(0,1)$ such that $g(x) \to \infty$ as $x \to 0$, then there is a convex function h on $(0,1)$ such that $h \leqslant g$ and $h(x) \to \infty$ as $x \to 0$. True or false? Is the problem changed if $(0,1)$ is replaced by $(0,\infty)$ and $x \to 0$ by $x \to \infty$?

Solution. (1) The first claim is true. Since $g(x) \to \infty$ as $x \to 0$, we are able to construct a sequence $\{x_n\}_{n\geqslant 1}$ such that $x_n \to 0$ and $g \geqslant n$ on $(0, x_n)$. For each n, we define h_n to be the function on $(0,1)$ whose graph consists of line segments connecting $(0,n)$, $(x_n, 0)$, and $(1,0)$. It is clear that each h_n is convex and bounded above by g. Finally, we define $h := \sup h_n$ (pointwise supremum). Since $h_n \geqslant n/2$ on $(0, x_n/2)$, it is clear that $h(x) \to \infty$ as $x \to \infty$, as required by our problem.

(2) The second claim is false. If $h(x) \to \infty$ as $x \to \infty$ and h is convex, then h needs to have at least linear growth rate. To see this, pick any $a < b$ with $h(a) < h(b)$. By convexity, we have

$$\frac{h(b+x) - h(b)}{x} \geqslant \frac{h(b) - h(a)}{b - a}$$

for all $x > 0$, so the slope of h after b is always at least the quotient of the RHS. Therefore any function with sublinear growth rate fails to meet this criterion. For example, no convex function bounds $g(x) := \sqrt{x}$ from below.

□

Problem 9

Suppose f is Lebesgue measurable on $(0,1)$ and not essentially bounded. By 4(e), $\|f\|_p \to \infty$ as $p \to \infty$. Can $\|f\|_p$ tend to ∞ arbitrarily slowly? More precisely, is it true that to every positive function Φ on $(0,\infty)$ such that $\Phi(p) \to \infty$, one can find an f such that $\|f\|_p \to \infty$ as $p \to \infty$, but $\|f\|_p \leqslant \Phi(p)$ for sufficiently large p's?

Solution. Yes. Since we are only interested in the tail of Φ and $\Phi(p) \to \infty$, WLOG we may assume that $\Phi \geqslant 1$. Let

$\{a_n\}_{n\geqslant 1}$ be a sequence with $a_n := \Phi(n)$ and define correspondingly (left-closed, right-open) intervals $\{E_n\}$ by

$$E_1 := [0, (2\Phi(1))^{-1}) \qquad E_{n+1} := [\sum_{k=1}^{n}(2\Phi(k))^{-k}, \sum_{k=1}^{n+1}(2\Phi(k))^{-k}).$$

(Basically the E_n's are disjoint, and E_n has length $(2\Phi(n))^{-n}$.) It follows immediately that

$$\left[\frac{a_n}{\Phi(p)}\right]^p m(E_n) = 2^{-n}\left(\frac{\Phi(n)^{p-n}}{\Phi(p)^p}\right) \leqslant \begin{cases} 2^{-n}\cdot\dfrac{1}{\Phi(p)^n} \leqslant 2^{-n} & \text{if } p < n \\ 2^{-n}\cdot\dfrac{\Phi(n)^p}{\Phi(p)^p} \leqslant 2^{-n} & \text{if } p \geqslant n. \end{cases}$$

Therefore, if we define $f := \sum_{n=1}^{\infty} a_n \chi_{E_n}$, then

$$\|f\|_p^p/\Phi(p)^p = \frac{1}{\Phi(p)^p}\int_0^1 \sum_{n=1}^{\infty} a_n^p \chi_{E_n}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{a_n^p}{\Phi(p)^p} m(E_n) \leqslant \sum_{n=1}^{\infty} 2^{-n} = 1,$$

so $\|f\|_p \leqslant \Phi(p)$ (for all p, assuming that $\Phi \geqslant 1$; alternatively, for large p's where $\Phi(p) \geqslant 1$). □

Problem 10

Suppose $f_n \in L^p(\mu)$ for $n = 1, 2, 3, \ldots$, and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. What relation exists between f and g?

Proof. It is clear that $\{f_n\}$ is Cauchy. By Theorem 3.12 there exists a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere; since the mother sequence converges to f, we have $\{f_{n_k}\} \to f$ a.e. as well. On the other hand $(\{f_n\} \supset \{f_{n_k}\}) \to g$ a.e., so $f = g$ a.e. □

Problem 11

Suppose $\mu(\Omega) = 1$ and suppose f, g are positive mesurable functions on Ω such that $fg \geqslant 1$. Prove that

$$\int_\Omega f\,\mathrm{d}\mu \cdot \int_\Omega g\,\mathrm{d}\mu \geqslant 1.$$

Proof. $fg \geqslant 1$ implies $\sqrt{fg} \geqslant 1$. Applying Hölder's inequality to $\sqrt{f}$ and $\sqrt{g}$ and conjugate pair $(1/2, 1/2)$ gives

$$1 = \int_\Omega \mathrm{d}\mu \leqslant \int_\Omega \sqrt{fg}\,\mathrm{d}\mu \leqslant \left\{\int_\Omega (\sqrt{f})^2\,\mathrm{d}\mu\right\}^{1/2}\left\{\int_\Omega (\sqrt{g})^2\,\mathrm{d}\mu\right\}^{1/2}$$

which, after taking square of both sides, completes the proof. □

Problem 12

Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If $A := \int_\Omega h\,\mathrm{d}\mu$, prove that

$$\sqrt{1 + A^2} \leqslant \int_\Omega \sqrt{1 + h^2}\,\mathrm{d}\mu \leqslant 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h can equality hold in either of the above inequalities. Prove them.

Proof. (1) If $A = \infty$ then the claim is trivial, as $h < \sqrt{1+h^2}$.

If $A \neq \infty$, the first $\leqslant$ is given by Jensen's inequality applied to the convex function $\Phi(x) := \sqrt{1+x^2}$. The second follows from noticing $\sqrt{1+h^2} \leqslant \sqrt{1+2h+h^2} = 1+h$ and integrating w.r.t. μ over Ω.

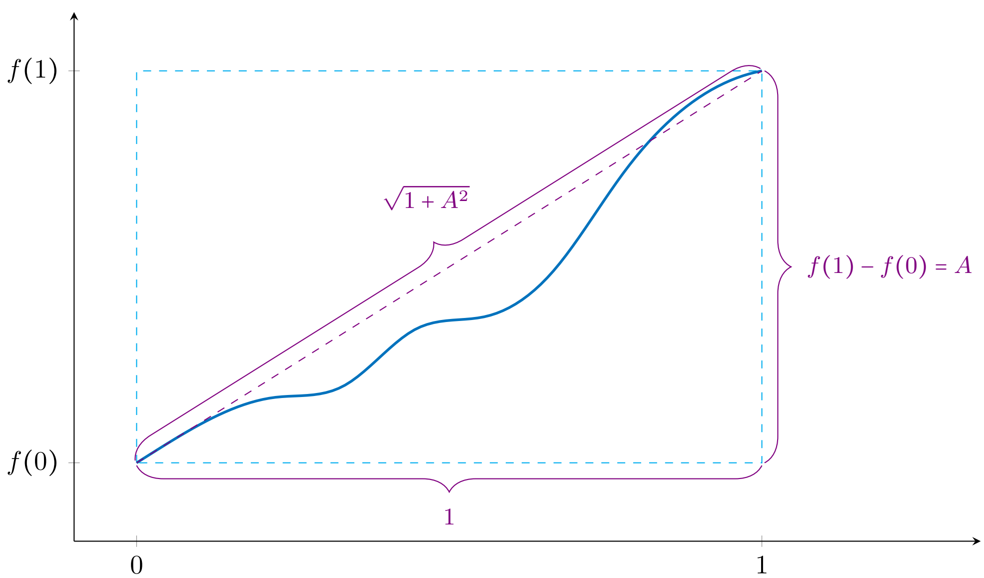
(2) Now let Ω be $[0,1]$ and μ be the Lebesgue measure m. Note that

$$A = \int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0),$$

and since $h = f'$, we can write

$$\int_{[0,1]} \sqrt{1+h^2}\,\mathrm{d}m = \int_0^1 \sqrt{1+f'(x)}\,\mathrm{d}x,$$

namely the arc length of the graph of f on $[0,1]$. See figure below.



Look at the purple right triangle. That $h \geqslant 0$ implies f is monotone increasing, and the inequalities state that the length of the blue curve (or the graph of any increasing function satisfying the boundary conditions) is bounded below by the length of the hypotenuse and above by the sum of the other two sides.

(3) For the $\leqslant$ to become $=$, we need Jensen's inequality to become equality, which is equivalent to requiring $\varphi(A) = \varphi(h(x))$ a.e., so $h \equiv A$ a.e. because φ is injective.

For the second $\leqslant$ to become $=$, we need $\sqrt{1+h^2} = 1+h$ a.e. so $h \equiv 0$ a.e. $\square$

Problem 13

Under what conditions on f and g does equality hold in the conclusions of Theorem 3.8 and 3.9? You may have to treat the cases $p = 1$ and $p = \infty$ separately.

Solution. (1) Hölder's, $1 < p < \infty$: re-examining Rudin's Theorem 3.5 equations (3) and (5), we see that the convexity of $\exp$ forces equality to take place if and only if $F = G$ almost everywhere. That is, if and only if one between f and g is 0 a.e. or

$$\frac{|f|}{A} = \frac{|g|}{B} \text{ a.e.} \iff \frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q} \text{ a.e.}$$

(Note that the ratio $\|f\|_p^p/\|g\|_q^q$ is in fact arbitrary: if f and g give us equality, then so do λf and g.)

(2) Hölder's, $p = \infty$ (or equivalently $q = 1$): re-examining Rudin's Theorem 3.8 equation (2), we see that the equality holds if and only if

$$|f(x)g(x)| = \|f\|_\infty |g(x)| \text{ a.e.},$$

so if $p = \infty$ then we need $|f(x)| = \|f\|_\infty$ almost everywhere.

(3) Minkowski's, $1 < p < \infty$: re-examining Rudin's Theorem 3.5 equation (9) and using part (1) [of this solution], we see that equality holds if and only if there are constants C_1, C_2 such that

$$|f|^q = C_1(|f+g|^{p-1})^q = C_1|f+g|^p \text{ a.e.} \qquad \text{and} \qquad |g|^p = C_2(|f+g|^{p-1})^q = C_2|f+g|^p \text{ a.e.}$$

Combining these two, we see that Minkowski's inequality attains equality if and only if

$$|f|^p = C|g|^p \text{ a.e. for some } C \implies |f| = \tilde{C}|g| \text{ a.e.}$$

(4) Minkowski's, $p = 1$: in this case we require $|f(x) + g(x)| = |f(x)| + |g(x)|$ almost everywhere, so it is necessary and condition to require that $|f(x)g(x) \geqslant 0|$ almost everywhere. (Both positive, both negative, or at least one is 0.)

(5) Minkowski's, $p = \infty$: by definition we need

$$\operatorname*{ess\,sup}_{x \in X} |f(x) + g(x)| = \operatorname*{ess\,sup}_{x \in X} |f(x)| + \operatorname*{ess\,sup}_{x \in X} |g(x)|. \qquad \square$$

Problem 14

Suppose $1 < p < \infty$, $f \in L^p = L^p((0,\infty))$ relative to the Lebesgue measure, and

$$F(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \qquad (0 < x < \infty).$$

(a) Prove **Hardy's inequality** $\|F\|_p \leqslant \dfrac{p}{p-1}\|f\|_p$ which shows that the mapping $f \mapsto F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $p/(p-1)$ cannot be made smaller.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Hint: for (a), assume first that $f \geqslant 0$ and $f \in C_c((0,\infty))$. Integration by part gives

$$\int_0^\infty F^p(x)\,\mathrm{d}x = -p\int_0^\infty F(x)^{p-1} x F'(x)\,\mathrm{d}x.$$

Note that $xF' = f - F$ and apply Hölder's inequality to $\int F^{p-1} f$. Then derive the general case. For (c), take $f(x) := x^{-1/p}$ on $[1, A]$ and 0 elsewhere for large A.

Proof. (a) Following the hint, first assume $f \geqslant 0$ and is compactly supported. Then,

$$\begin{aligned}
\|F\|_p^p = \int_0^\infty F^p(x)\,\mathrm{d}x &= -p\int_0^\infty F(x)^{p-1} x F'(x)\,\mathrm{d}x \\
&= -p\int_0^\infty F(x)^{p-1}\cdot(f(x) - F(x))\,\mathrm{d}x \\
&= -p\int_0^\infty F(x)^{p-1} f(x)\,\mathrm{d}x + p\|F\|_p^p \\
[\text{Hölder's}] &\geqslant -p\left\{\int_0^\infty (F(x)^{p-1})^{p/(p-1)}\,\mathrm{d}x\right\}^{(p-1)/p}\left\{\int_0^\infty f(x)^p\,\mathrm{d}x\right\}^{1/p} + p\|F\|_p^p \\
&= -p\|F\|_p^{p-1}\|f\|_p + p\|F\|_p^p. \qquad (\Delta)
\end{aligned}$$

Rearranging gives

$$(p-1)\|F\|_p^p \leqslant p\|F\|_p^{p-1}\|f\|_p \implies \|F\|_p \leqslant \frac{p}{p-1}\|f\|_p.$$

For the more general case, since the sign of f has no impact on $\|\cdot\|_p$, we may still assume that $f \geqslant 0$. By Rudin's Theorem 3.14, $C_c((0,\infty))$ is dense in $L^p((0,\infty))$, so for any $f \in L^p$ there exists a sequence $\{f_n\}$ of compactly supported functions such that $\|f_n - f\|_p \to 0$. If we define F_n's and F accordingly using the integral definition, then it follows that $F_n \to F$ pointwise:

$$\begin{aligned}
|F_n(x) - F(x)| &= \frac{1}{x}\left|\int_0^x f_n(t) - f(t)\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{x}\int_0^x |f_n(t) - f(t)|\,\mathrm{d}t \\
&\leqslant \left\{\int_0^x |f_n(t) - f(t)|^p\,\mathrm{d}t\right\}^{1/p}\left\{\int_0^x \mathrm{d}t\right\}^{1-1/p} \\
&= \|f_n - f\|_p \cdot x^{1-1/p} \to 0
\end{aligned}$$

for any (fixed) x as $n \to \infty$. Therefore,

$$\begin{aligned}
\|F\|_p^p = \int_0^\infty F(x)^p\,\mathrm{d}x = \int_0^\infty \liminf_{n\to\infty} F_n(x)^p\,\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty}\int_0^\infty F_n(x)^p\,\mathrm{d}x && [\text{Fatou's lemma}] \\
&\leqslant \liminf_{n\to\infty}\left(\frac{p}{p-1}\right)^p \|f_n\|_p^p && [\text{since } f_n \in C_c((0,\infty))] \\
&= \left(\frac{p}{p-1}\right)^p \|f\|_p^p.
\end{aligned}$$

Taking the p^{th} root of both sides, we recover Hardy's inequality.

(b) For convenience, when context is clear, we denote $1 - 1/p$ by $1/q$ from now on.

Assume f is not 0 a.e. Since $\|f\|_p = \||f|\|_p$, we can WLOG assume that f is nonnegative. (It is impossible that for f to change sign if equality is attained: if so, define

$$\tilde{F}(x) := \frac{1}{x}\int_0^x |f(t)|\,\mathrm{d}t \implies \|\tilde{F}\|_p > \|F\|_p = \frac{p}{p-1}\||f|\|_p,$$

contradiction. Hence our "WLOG" is well-defined.) Since the mapping $f \mapsto F$ is continuous (by (a)), (Δ) in fact holds for any f, not just the positive, compactly supported ones. (Recall we can approximate f in $\|\cdot\|_p$ by a sequence of compactly supported functions $\{f_n\}$.)

In order to make Hardy's inequality an equality, the Hölder step needs to attain =; since neither f nor F^{p-1} is 0 a.e., by the previous problem we must have

$$\frac{|f|^p}{|F^{p-1}|^q} = \frac{|f|^p}{|F|^p} = \text{ some constant a.e.}$$

Thus f/F equals some constant $C > 0$ a.e. Since

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t = \frac{1}{x}\int_0^x CF(t)\,\mathrm{d}t,$$

from the first = we see that F is continuous, and the second further implies that F is differentiable. This in turn implies f is differentiable. Thus f'/F' also equals C a.e. Recall (the hint says) $xF' = f - F$, so

$$xCf' = f - Cf \implies xf' = (1/C - 1)f.$$

This is a differential equation whose solution suggests

$$f(x) = kx^{(1/C-1)}$$

for some constant k. However, $1/C - 1 > -1$ so f cannot be in L^p. This shows that the only functions that turn Hardy's inequality into equality are those that are 0 almost everywhere.

(c) I don't understand why we need to cook up an example involving $x^{-1/p}$, given that part (b) already established examples where $p/(p-1)$ is attained.

(d) If $f > 0$ and $f \in L^1$, then there exists some interval $[a, b]$ on which $f \geqslant \epsilon > 0$. Then, for all $x \geqslant b$,

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \geqslant \frac{1}{x}\int_a^b \epsilon\,\mathrm{d}t = \frac{\epsilon(b-a)}{x}.$$

Since $\int_0^\infty \frac{1}{x}\,\mathrm{d}x = \infty$, we conclude that $F \notin L^1$. □

Problem 15

Suppose $\{a_n\}$ is a sequence of positive numbers. Prove that

$$\sum_{N=1}^\infty \left(\frac{1}{N}\sum_{n=1}^N a_n\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$$

if $1 < p < \infty$. *Hint: if $a_n \geqslant a_{n+1}$, the result can be made to follow from Problem 14. The general case follows.*

Proof. Notice that this is a "discrete" version of Problem 14. We define

$$f(x) := \sum_{n=1}^{\infty} a_n \chi_{(n-1,n]} \qquad \text{on } (0,\infty).$$

Immediately we see that $f \in L^p$ if and only if $\sum_{n=1}^{\infty} a_n^p$ is finite. If $f \notin L^p$ then both sides are ∞ and the claim holds trivially. If $f \in L^p$, the corresponding F is defined by

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t = \frac{1}{x}\left(\sum_{n=1}^{\lfloor x\rfloor} a_n + (x - \lfloor x\rfloor)a_{\lfloor x\rfloor+1}\right).$$

By the previous problem, $\|F\|_p \leqslant \|f\|_p \cdot p/(p-1)$. If we assume $a_n \geqslant a_{n+1}$, then

$$\begin{aligned}\frac{1}{x}\left(\sum_{n=1}^{\lfloor x\rfloor} a_n + (x-\lfloor x\rfloor)a_{\lfloor x\rfloor+1}\right) &= \frac{1}{x}\left(\sum_{n=1}^{\lfloor x\rfloor} a_n - \lfloor x\rfloor\, a_{\lfloor x\rfloor+1}\right) + a_{\lfloor x\rfloor+1}\\ &\geqslant \frac{1}{x}\sum_{n=1}^{\lfloor x\rfloor} a_n + a_{\lfloor x\rfloor+1} \geqslant \frac{1}{\lfloor x\rfloor+1}\sum_{x=1}^{\lceil x\rceil+1} a_n\end{aligned}$$

so $F(\lceil x\rceil) \leqslant F(x)$. Therefore,

$$\begin{aligned}\sum_{N=1}^{\infty}\left(\frac{1}{N}\sum_{n=1}^{N} a_n\right)^p &= \sum_{n=1}^{\infty}\int_{n-1}^{n} F(\lfloor x\rfloor)^p\,\mathrm{d}x = \int_0^{\infty} F(\lceil x\rceil)^p\,\mathrm{d}x\\ &\leqslant \int_0^{\infty} F(x)^p\,\mathrm{d}x = \|F\|_p^p \leqslant \left(\frac{p}{p-1}\right)^p \|f\|_p^p = \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.\end{aligned}$$

For the general case, since $f \in L^p$, $\sum_{n=1}^{\infty} a_n^p$ converges (absolutely), so any rearrangement will converge to the same sum, so the RHS remains the same. The LHS, however, is maximal if $\{a_n\}$ is decreasing, for by doing so we count the large elements more times, thereby giving them "more weight". □

Problem 16

Prove **Egoroff's theorem**: if $\mu(X) < \infty$, if $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X, and if $\epsilon > 0$, then there is a measurable set $E \subset X$ with $\mu(X - E) < \epsilon$, on which $\{f_n\}$ converges uniformly. (The conclusion is that by redefining the f_n on a set of arbitrarily small measure, we can convert a pointwise convergent sequence to a uniformly convergent one.)

Hint: put

$$S(n,k) := \bigcap_{i,j>n} \{x : |f_i(x) - f_j(x)| < 1/k\}.$$

Show that $\mu(S(n,k)) \to \mu(X)$ as $n \to \infty$ for each k, and hence there is a suitably increasing sequence $\{n_k\}$ such that $E = \bigcap S(n_k,k)$ has the desired property.

Show that the theorem does not extend to σ-finite spaces. Also show that the theorem does extend, with essentially the same proof, to the situation in which the sequence $\{f_n\}$ is replaced by a family $\{f_t\}_{t\in\mathbb{R}+}$; the assumptions are now that for all $x \in X$, (i) $\lim_{t\to\infty} f_t(x) = f(x)$ and (ii) $t \mapsto f_t(x)$ is continuous.

Proof. (1) For a fixed k, $S(1,k) \subset S(2,k) \subset \ldots$, and $\bigcup_{n=1}^{\infty} S(n,k) = \lim_{j\to\infty} \bigcup_{n=1}^{j} S(n,k) = \lim_{j\to\infty} S(j,k) = X$, so by Rudin's Theorem 1.19(d), $\mu(S(n,k)) \to \mu(X)$ as $n \to \infty$.

For each k, we can pick a corresponding n_k such that $\mu(S(n_k,k)) > \mu(X) - \epsilon 2^{-k}$, or equivalently $\mu(S(n_k,k)^c) < \epsilon 2^{-k}$. In other words, for this k, the following set has measure $< \epsilon 2^{-k}$:

$$\bigcup_{i,j>n_k} \{x : |f_i(x) - f_j(x)| \geqslant 1/k\}.$$

Finally we define $E^c := \bigcup_{k=1}^{\infty} S(n_k,k)^c$ so $\mu(E^c) < \epsilon$. We claim that E is the set we are looking for: for all $i,j > n_k$, $|f_i(x) - f_j(x)| < 1/k$ on E for all $x \in X$, so pointwise convergence upgrades to uniform convergence on E, as claimed.

(2) (The main problem with a σ-finite space is that Rudin's Theorem 1.19(e) requires a finite measure.) To derive a contradiction, for a fixed k, we can let $S(n,k)^c$ be $[n,\infty)$ so that 1.19(e) fails:

$$\lim_{n\to\infty} \mu(S(n,k)^c) = \infty \neq 0 = \mu(\bigcap_{n=1}^{\infty} S(n,k)^c).$$

If so, there is no guarantee that we can find sufficiently large n_k satisfying $\mu(S(n_k,k)) > \mu(X) - \epsilon 2^{-k}$.

One such example is if we let $f_n := \chi_{[n-1,n)}$ on $[0,\infty)$. Obviously $[0,\infty)$ is σ-finite; it is also clear that $\{f_n\} \to$ the zero function pointwise everywhere. Meanwhile, $S(n,1) = [0,n)$ so $S(n,1)^c = [n,\infty)$, and so Rudin's 1,19(e) does not apply. We see that $f_n \to f$ uniformly on *no* subset of $[0,\infty)$!

(3) Since the set of positive rationals is dense in $(0,\infty)$, (1) gives the claim for $\{f_n\}_{n\in\mathbb{Q}+}$, and the continuity assumptions allows us to extend this to $\{f_n\}_{n\in\mathbb{R}+}$.

□

Problem 17

(a) If $0 < p < \infty$, put $\gamma_p := \max(1, 2^{p-1})$. Prove that

$$|\alpha - \beta|^p \leqslant \gamma_p(|\alpha|^p + |\beta|^p)$$

for arbitrary complex numbers α and β.

(b) Suppose μ is a positive measure on X, $0 < p < \infty$, $f \in L^p(\mu)$, $f_n \in L^p(\mu)$, $f_n(x) \to f(x)$ a.e., and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Show that $\lim \|f - f_n\|_p = 0$ by completing the two proofs sketched below:

(i) By Egoroff's theorem, $X = A \cup B$ in such a way that $\int_A |f|^p < \epsilon$, $\mu(B) < \infty$, and $f_n \to f$ uniformly on B. Fatou's lemma, applied to $\int_B |f_n|^p$, leads to $\limsup \int_A |f_n|^p \, d\mu \leqslant \epsilon$.

(ii) Put $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$ and use Fatou's lemma as in the proof of Lebesgue's DCT.

(c) Show that the conclusion of (b) is false if the hypothesis $\|f_n\|_p \to \|f\|_p$ is omitted, even if $\mu(X) < \infty$.

Proof. (a) If $p > 1$ then

$$|\alpha - \beta|^p \leqslant (|\alpha| + |\beta|)^p = 2^p \left(\frac{|\alpha| + |\beta|}{2}\right)^p \leqslant 2^{p-1}(|\alpha|^p + |\beta|^p) \tag{1}$$

since x^p is convex on $[0, \infty)$.

If $p \leqslant 1$, put $r = 1 - p$. Then

$$(|\alpha| + |\beta|)^p = (|\alpha| + |\beta|)^{1-r} = |\alpha|(|\alpha| + |\beta|)^{-r} + |\beta|(|\alpha| + |\beta|)^{-r} \leqslant |\alpha|^{1-r} + |\beta|^{1-r} = |\alpha|^p + |\beta|^p. \tag{2}$$

Finally, combining (1) and (2) and using the fact that $|\alpha - \beta| \leqslant |\alpha| + |\beta|$, we complete the proof.

(b) (i) We first prove two claims, after which (b).(i) is almost an immediate consequence.

Claim 1. If $g \in L^1$ and $\epsilon > 0$, then there exists a set E of *finite* measure such that $\int_{X-E} |g| \, \mathrm{d}\mu < \epsilon$.

Proof of Claim 1. We can WLOG assume $g \geqslant 0$. Since g is measurable, by Rudin's Theorem 1.17, there exists a sequence $\{s_n\}$ of increasing simple functions converging pointwise to g. By Lebesgue's DCT, the integrals converge too, so there exists a sufficiently large k such that

$$\int_X |g - s_k| \, \mathrm{d}\mu < \epsilon.$$

Clearly, as $s_k \leqslant g$, it is in L^1; since it takes the form $\sum_{i=1}^{n_k} \alpha_i \chi_{E_i}$, it follows from the L^1-integrability that each $\mu(E_i)$ is finite. If we let $E := \bigcup_{i\in=1}^{n_k} E_i$, then $\mu(E) < \infty$. Also, s_k vanishes on $X - E$. We claim that E is our desired set:

$$\int_{X-E} |g| \, \mathrm{d}\mu = \int_{X-E} |g - s_k| \, \mathrm{d}\mu + \int_{X-E} s_k \, \mathrm{d}\mu \leqslant \int_X |g - s_k| \, \mathrm{d}\mu + \int_{X-E} 0 \, \mathrm{d}\mu < \epsilon.$$

END OF PROOF OF CLAIM 1.

Claim 2. If $f \in L^p$, then given $\epsilon > 0$ there exists $\delta > 0$ such that if E is measurable and $\mu(E) < \delta$ then $\int_E |f|^p \, \mathrm{d}\mu < \epsilon$.

Proof of Claim 2. It is clear that $|f|$ must be essentially bounded (otherwise $f \notin L^p$) by, say M. Then if we set $\delta := \epsilon / M^p$ and if $\mu(E) < \delta$,

$$\int_E |f|^p \, \mathrm{d}\mu < \int_E M^p \, \mathrm{d}\mu = \frac{M^p \epsilon}{M^p} = \epsilon.$$

END OF PROOF OF CLAIM 2.

Back to the main proof: Since $f \in L^p$, $|f|^p \in L^1$. Using Claim 1, we can find a set $\tilde{B}$ of finite measure such that

$$\int_{X-\tilde{B}} |f|^p \, \mathrm{d}\mu < \frac{\epsilon}{2}. \tag{3}$$

Next, Claim 2 gives a δ such that if $\mu(E) < \delta$ then

$$\int_E |f|^p \, \mathrm{d}\mu < \frac{\epsilon}{2}. \tag{4}$$

Finally, we construct A and B. Using Egoroff's theorem, corresponding to this δ, there exists $B \subset \tilde{B}$ such that $\mu(\tilde{B} - B) < \delta$ and $\{f_n\}$ converges uniformly on B. We define $A := (\tilde{B} - B) \cup E$; combining (3) and (4), we have

$$\int_A |f|^p \, \mathrm{d}\mu = \int_{X-\tilde{B}} |f|^p \, \mathrm{d}\mu + \int_E |f|^p \, \mathrm{d}\mu < \epsilon. \tag{5}$$

Now, applying Fatou's lemma to $|f_n|^p$ gives

$$\int_B |f|^p \, \mathrm{d}\mu = \int_B \liminf_{n\to\infty} |f_n|^p \, \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_B |f_n|^p \, \mathrm{d}\mu,$$

so

$$\begin{aligned}
\limsup_{n\to\infty} \int_A |f_n|^p \, \mathrm{d}\mu &= \limsup_{n\to\infty} \left\{ \int_X |f_n|^p \, \mathrm{d}\mu - \int_B |f_n|^p \, \mathrm{d}\mu \right\} \\
&\leqslant \limsup_{n\to\infty} \int_X |f_n|^p \, \mathrm{d}\mu + \limsup_{n\to\infty} \left\{ - \int_B |f_n|^p \, \mathrm{d}\mu \right\} \\
&= \int_X |f|^p \, \mathrm{d}\mu - \liminf_{n\to\infty} \int_B |f_n|^p \, \mathrm{d}\mu && [\|f_n\|_p \to \|f\|_p] \\
&\geqslant \int_X |f|^p \, \mathrm{d}\mu - \int_B \liminf_{n\to\infty} |f_n|^p \, \mathrm{d}\mu \\
&= \int_X |f|^p \, \mathrm{d}\mu - \int_B |f|^p \, \mathrm{d}\mu = \int_A |f|^p \, \mathrm{d}\mu < \epsilon. && [\text{by (5)}] \qquad (6)
\end{aligned}$$

Finally, since $\mu(B) < \infty$ and $f_n \to f$ uniformly on B, $\int_B |f_n - f|^p \, \mathrm{d}\mu$ can be made arbitrarily small, say $< \epsilon$ for example; using part (a) on $|f_n - f|^p$, we see

$$\begin{aligned}
\limsup_{n\to\infty} \int_X |f_n - f|^p \, \mathrm{d}\mu &\leqslant \limsup_{n\to\infty} \int_A |f_n - f|^p \, \mathrm{d}\mu + \limsup_{n\to\infty} \int_B |f_n - f|^p \, \mathrm{d}\mu \\
&< \limsup_{n\to\infty} \int_A \gamma_p(|f_n|^p + |f|^p) \, \mathrm{d}\mu + \epsilon && [\text{(a) and above}] \\
&\leqslant \gamma_p \left\{ \limsup_{n\to\infty} \int_A |f_n|^p \, \mathrm{d}\mu + \limsup_{n\to\infty} \int_A |f|^p \, \mathrm{d}\mu \right\} + \epsilon && [\text{splitting limsup}] \\
&< \gamma_p(\epsilon + \epsilon) + \epsilon = C\epsilon,
\end{aligned}$$

so indeed $\|f_n - f\|_p \to 0$ as $n \to \infty$, and we are done.

(ii) Second proof: following the hint, define

$$h_n := \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p.$$

By (a), each $h_n \geqslant 0$. Since $f_n \to f$ a.e., $h_n \to 2\gamma_p|f|^p$ a.e., so

$$\begin{aligned}
\int_X 2\gamma_p |f|^p \, \mathrm{d}\mu = \int_X \lim_{n\to\infty} h_n \, \mathrm{d}\mu &= \int_X \liminf_{n\to\infty} h_n \, \mathrm{d}\mu \leqslant \liminf_{n\to\infty} \int_X h_n \, \mathrm{d}\mu \\
&= \liminf_{n\to\infty} \left\{ \int_X \gamma_p(|f|^p + |f_n|^p) \, \mathrm{d}\mu - \int_X |f - f_n|^p \, \mathrm{d}\mu \right\} \\
&= \int_X 2\gamma_p |f|^p \, \mathrm{d}\mu - \limsup_{n\to\infty} \int_X |f - f_n|^p \, \mathrm{d}\mu.
\end{aligned} \tag{7}$$

(The last step, (7), follows from the fact that $\liminf(a_n + b_n) = \liminf a_n + \liminf b_n$ if $\{a_n\}$ converges and $\{b_n\}$ is bounded. The proof of this can be easily derived by constructing a convergent subsequence $\{b_{n_k}\}$ using Bolzano-Weierstraß. In this case the first integral converges and the second is bounded, as $f, f_n \in L^p$.)

Since $\int_X 2\gamma_p |f|^p \, d\mu$ is finite, we may subtract it from both sides and obtain

$$0 \leqslant -\limsup_{n\to\infty} \int_X |f - f_n|^p \, d\mu.$$

Of course, the RHS is nonpositive, so $\leqslant$ becomes $=$, and indeed

$$\lim_{n\to\infty} \|f_n - f\|_p = \limsup_{n\to\infty} \|f_n - f\|_p = 0.$$

(iii) Consider $p = 1$, $X = [0,1]$, $\mu = m$, $f \equiv 0$, and $f_n = n\chi_{[0,1/n]}$. Then $f_n \to f$ a.e. (for all points but 0), $\mu(X) = 1$, but the Lebesgue integral of f_n is always 1, whereas that of f is 0.

□

Problem 18

let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to **converge in measure** to the measurable function f if for every $\epsilon > 0$, there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon \qquad \text{for all } n \geqslant N.$$

Assume $\mu(X) < \infty$ and prove the following statements.

(a) If $f_n(x) \to f(x)$ a.e., then $f_n \to f$ in measure.

(b) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure; here $1 \leqslant p \leqslant \infty$.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to f a.e.

Investigate the converses of (a) and (b). What happens to (a), (b), and (c) if $\mu(X) = \infty$, for instance, if μ is the Lebesgue measure on $\mathbb{R}$?

Proof. (a) Fix ϵ and define $E_N := \{x \in X : |f_n(x) - f(x)| > \epsilon \text{ for } \textit{some } n \geqslant N\}$. It is clear that $E_1 \supset E_2 \supset \dots$ and $\mu(E_1) \leqslant \mu(X) < \infty$. Therefore, by Rudin's Theorem 1.19(e), $\mu(E_n) \to \mu(\bigcap_{n=1}^{\infty} E_n)$. The infinite intersection is a null set because $f_n \to f$ almost everywhere, so $\mu(E_n) \to 0$. In particular, there exists a sufficiently large N such that $\mu(E_N) < \epsilon$. This implies that, for all $n \geqslant N$,

$$\{x : |f_n(x) - f(x)| > \epsilon\} \subset E_n,$$

so taking the union of all such n's, the inclusion still holds. Finally, taking the measure of both sides gives us the claim.

(b) Let $\epsilon > 0$ be given.

If $p = \infty$, then $\|f_n - f\|_\infty \to 0$ means that there exists a sufficiently large $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \epsilon$ whenever $n \geqslant N$, so in these cases $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) = 0$, completing the proof.

If $p < \infty$, we define $E_n := \{x : |f_n(x) - f(x)| > \epsilon\}$. Then,

$$\|f_n - f\|_p^p = \int_X |f_n(x) - f(x)|^p \, \mathrm{d}\mu \geqslant \int_{E_n} |f_n(x) - f(x)|^p \, \mathrm{d} > \epsilon^p \mu(E_n).$$

Since $\|f_n - f\|_p \to 0$, we have $\mu(E_n) = 0$, so $\mu(E_n) < \epsilon$ for all $n \geqslant$ some sufficiently large N.

(c) By assumption, there exists a sequence of numbers $\{n_k\}$ such that

$$\mu(\{x : |f_n(x) - f(x)| > 2^{-k}\}) < 2^{-k} \qquad \text{for all } n \geqslant n_k.$$

We claim that $\{f_{n_k}\}$ is the subsequence we are looking for. To this end, define

$$E_k := \{x : |f_{n_k}(x) - f(x)| > 2^{-k}\}.$$

Finally, define $E := \bigcap_{m=1}^{\infty} \bigcup_{k \geqslant m}^{\infty} E_k$. It follows that if $x \notin E$, then $x \notin \bigcup_{k \geqslant m}^{\infty} E_k$, for some m, and in particular $x \notin E_k$ for *all* $k \geqslant m$. This means that

$$|f_{n_k}(x) - f(x)| \leqslant 2^{-k}$$

for all $k \geqslant m$, so $f_{n_k}(x)$ converges to $f(x)$. It remains to show that E is a set of measure zero. It is clear that $\mu(E_k) < 2^{-k} = \mu(\{x : |f_{n_k}(x) - f(x)| > 2^{-k}\}) \leqslant \mu(\{x : |f_n(x) - f(x)| > 2^{-k} \text{ for all } n \geqslant n_k\}) < 2^{-k}$, so $\bigcup_{k \geqslant 1}^{\infty} E_k$ has finite measure, and Rudin's Theorem 1.19(e) gives

$$\mu(E) = \lim_{m \to \infty} \mu(\bigcup_{k \geqslant m}^{\infty} E_k) = \lim_{m \to \infty} \frac{1}{2^{m-1}} = 0.$$

This completes the proof.

(d) The converse of (a) does not hold: for example consider

$$f_1 = \chi_{[0,1]}$$

$$f_2 = \chi_{[0,1/2]} \qquad f_3 = \chi_{[1/2,1]}$$

$$f_4 = \chi_{[0,1/3]} \qquad f_5 = \chi_{[1/3,2/3]} \qquad f_6 = \chi_{[2/3,1]}$$

in $L^1([0,1])$ and so on. They converge to the zero function in measure, as

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

if n is so large that the length of the interval on which $f_n \equiv 1$ is less than ϵ. However, it is clear that $\{f_n\}$ does not converge pointwise to the zero function at all — for each x, $f_n(x) = 1$ for infinitely many n's.

The converse of (b) does not hold, either. Consider $f_n : n\chi_{(0,1/n)}$ and let $f \equiv 0$, all of which are in $L^1((0,1))$. Then $f_n \to f$ in measure because they disagree only on an interval of length $1/n$. However, $\|f_n - f\|_1$ is always 1.

Finally, if we let $X = \mathbb{R}$ and $\mu = m$, then (a) and (b) still hold, as their proof impose no restriction on $\mu(X)$. (3), however, would fail: consider a modified example on $L^1((0,\infty))$:

$$f_1 = \chi_{[0,1]}$$

$$f_2 = \chi_{[1,1+1/2]} \qquad f_3 = \chi_{[1+1/2,2]}$$

$$f_3 = \chi_{[2+1/3]} \qquad f_5 = \chi_{[2+1/3,2+2/3]} \qquad f_6 = \chi_{[[2+2/3,3]}$$

and so on. $f_n \to f \equiv 0$ in measure, since each f_n differ from f on an interval with length $1/k$ for some k. However, $\{f_n\}$ clearly cannot admit any subsequence that converges to f a.e. □

Problem 19

Define the *essential range* of a function $f \in L^\infty(\mu)$ to be the set R_f consisting of all complex numbers w such that

$$\mu(\{x : |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that R_f is compact. What relation exists between the set R_f and the number $\|f\|_\infty$? Let A_f be the set of all averages

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu$$

where $E \in \mathfrak{M}$ and $\mu(E) > 0$. What relations exist between A_f and R_f? Is A_f always closed? Are there measures μ such that A_f is convex for every $f \in L^\infty(\mu)$? Are there measures μ such that A_f fails to be convex for some $f \in L^\infty(\mu)$?
How are these results affected if $L^\infty(\mu)$ is replaced by $L^1(\mu)$, for instance?

Proof. (1) To show compactness of R_f, it suffices to show that it's closed and bounded (since Heine-Borel applies to $\mathbb{C}$). Since $|f| \leqslant \|f\|_\infty$ almost everywhere, R_f is bounded by the closed disk of radius $\|f\|_\infty$.

It remains to show that R_f is closed. Let $\{z_n\} \subset R_f$ be a sequence of complex numbers converging to $z \in \mathbb{C}$. Let $\epsilon > 0$ be given; there corresponds an N such that $|z_n - z| < \epsilon/2$ whenever $n \geqslant N$. Note that

$$|f(x) - z| \leqslant |f(x) - z_n| + |z_n - z| < |f(x) - z_n| + \frac{\epsilon}{2}.$$

Since $z_n \in R_f$, the set $\{x : |f(x) - z_n| < \epsilon/2\}$ has positive measure. Note that if x is in this set then $|f(x) - z| < \epsilon/2 + \epsilon/2 = \epsilon$, so

$$\{x : |f(x) - z_n| < \epsilon/2\} \subset \{x : |f(x) - z| < \epsilon\} \implies \mu(\{x : |f(x) - z| < \epsilon\}) > 0.$$

Since ϵ is arbitrary, we conclude $z \in R_f$, so R_f is closed and bounded; Heine-Borel implies it is compact.

Intuitively, $\|f\|_\infty = \max\{|w| : w \in R_f\}$. (The $\max$ is attained int his situation. Or maybe I should have chosen $\sup$ instead for clarify?)

(2) We claim that $R_f \subset \overline{A_f}$. Suppose $w \in R_f$. Let $\epsilon = 1/n$ (where $n \in \mathbb{N}$ increases). By definition of R_f, $\mu(E_n) := \mu(\{x \in X : |f(x) - w| < 1/n\}) > 0$. If each E_n has infinite measure, then $f \equiv w$ almost everywhere so

$$\frac{1}{\mu(E_n)} \int_{E_n} f \, \mathrm{d}\mu = w \in A_f$$

for all n, and thus $w \in \overline{A_f}$. Therefore, late enough E_n's must satisfy $\mu(E_n) < \infty$ (notice that $E_1 \supset E_2 \supset \ldots$). Therefore,

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f \, \mathrm{d}\mu - w \right| \leqslant \frac{1}{\mu(E_n)} \int_{E_n} |f - w| \, \mathrm{d}\mu < \frac{1}{n},$$

so again we see that there exists a sequence of form $\int_{E_n} f \, \mathrm{d}\mu / \mu(E_n)$ that converges to w. Thus $w \in \overline{A_f}$.

(3) Consider $f(x) := 1/n$ on $(1, \infty)$. It is clear that by considering intervals of form $(0, n)$,

$$\frac{1}{n}\int_1^n \frac{1}{x}\,\mathrm{d}x = \frac{\log n}{n} \to 0 \qquad \text{as } n \to \infty,$$

but A_f does not contain 0, as the integral of $1/x$ on any interval is positive.

(4) A trivial example where A_f is convex is by letting X be a singleton, resulting in A_f and R_f both being singletons (and of course singletons are convex).

(5) An example that makes A_f not convex: let $X = \{0, 1\}$, μ be the counting measure, and $f(x) = x$. Then $A_f = \{0, 1/2, 1\}$, which is not convex.

(6) If we replace $L^\infty(\mu)$ by $L^1(\mu)$, we can change everything. For example, the function

$$f(x) := \begin{cases} \dfrac{1}{\sqrt{x}} & 0 < x < 1 \\ 0 & x \geqslant 1 \end{cases}$$

is not essentially bounded but indeed L^1, with $\|f\|_1 = 2$. It follows that R_f and $\overline{A_f}$ are both unbounded, and so is A_f. □

Problem 20

Suppose φ is a real function on $\mathbb{R}$ such that

$$\varphi\left(\int_0^1 f(x)\,\mathrm{d}x\right) \leqslant \int_0^1 \varphi(f(x))\,\mathrm{d}x$$

for all real bounded measurable f. Prove that φ is then convex.

Proof. Let $a < b$ be two real numbers and $\lambda \in [0, 1]$. We want to show that $\varphi((1-\lambda)a + \lambda b) \leqslant (1-\lambda)\varphi(a) + \lambda\varphi(b)$. Consider the function

$$f(x) := a\chi_{[\lambda,1]}(x) + b\chi_{[0,\lambda]}(x).$$

We claim that this f shows the convexity of φ:

$$\begin{aligned}
\varphi\left(\int_0^1 f(x)\,\mathrm{d}x\right) &= \varphi\left(\int_0^1 a\chi_{[\lambda,1]}(x)\,\mathrm{d}x + \int_0^1 b\chi_{[0,\lambda]}(x)\,\mathrm{d}x\right) \\
&= \varphi((1-\lambda)a + \lambda b) \\
&\leqslant \int_0^1 \varphi(a\chi_{[\lambda,1]}(x) + b\chi_{[0,\lambda]}(x))\,\mathrm{d}x \\
&= \int_0^\lambda \varphi(b\chi_{[0,\lambda]}(x))\,\mathrm{d}x + \int_\lambda^1 \varphi(a\chi_{[\lambda,1]}(x))\,\mathrm{d}x \\
&= \lambda\varphi(b) + (1-\lambda)\varphi(a).
\end{aligned}$$

□

Problem 23

Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $\|f\|_\infty > 0$, and

$$\alpha_n = \int_X |f|^n\,\mathrm{d}\mu \qquad (n = 1, 2, 3, \ldots).$$

Prove that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty$.

Proof. WLOG assume $f \geqslant 0$. (That $\|f\|_\infty > 0$ already implies that each $a_n \neq 0$ so division is well-defined.) Immediately from the fact that $|f(x)| \leqslant \|f\|_\infty$ a.e., we see

$$a_{n+1} = \int_X |f|^{n+1}\,\mathrm{d}\mu \leqslant \int_X |f|^n \|f\|_\infty\,\mathrm{d}\mu = a_n\|f\|_\infty,$$

so

$$\frac{a_{n+1}}{a_n} \leqslant \|f\|_\infty \implies \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \leqslant \|f\|_\infty. \tag{1}$$

On the other hand, using Hölder's inequality on the conjugate pair $((n+1)/n, n+1)$ gives

$$\begin{aligned} a_n = \int_X |f|^n\,\mathrm{d}\mu &\leqslant \left\{\int_X (|f|^n)^{(n+1)/n}\,\mathrm{d}\mu\right\}^{n/(n+1)} \left\{\int_X \mathrm{d}\mu\right\}^{1/(n+1)} \\ &= \|f\|_{n+1}^n \mu(X)^{1/(n+1)} = a_{n+1}^{n/(n+1)} \mu(X)^{1/(n+1)}. \end{aligned}$$

Hence

$$\frac{a_{n+1}}{a_n} \geqslant \frac{\|f\|_{n+1}^{n+1}}{\|f\|_{n+1}^n \mu(X)^{1/(n+1)}} = \|f\|_{n+1}/\mu(X)^{1/(n+1)}.$$

Using Problem 4(e) and letting $n \to \infty$, we see that the RHS tends to $\|f\|_\infty/1 = \|f\|_\infty$, so

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \geqslant \|f\|_\infty. \tag{2}$$

Combining (1) and (2), we recover the desired equality. □

Problem 24

Suppose μ is a positive measure and $f, g \in L^p(\mu)$.

(a) If $0 < p < 1$, prove that

$$\int \big||f|^p - |g|^p\big|\,\mathrm{d}\mu \leqslant \int |f-g|^p\,\mathrm{d}\mu.$$

that $\Delta(f,g) := \int |f-g|^p\,\mathrm{d}\mu$ defines a metric on $L^p(\mu)$, and that the resulting metric space is complete.

(b) If $1 \leqslant p < \infty$ and $\|f\|_p \leqslant R$, $\|g\|_p \leqslant R$, prove that

$$\int \big||f|^p - |g|^p\big|\,\mathrm{d}\mu \leqslant 2pR^{p_1}\|f-g\|_p.$$

(a) and (b) establish the continuity of the mapping $f \mapsto |f|^p$ that carries $L^p(\mu)$ into $L^1(\mu)$.

Hint: first prove that for $x, y \geqslant 0$

$$|x^p - y^p| \leqslant \begin{cases} |x-y|^p & 0 < p < 1 \\ p|x-y|(x^{p-1} + y^{p-1}) & 1 \leqslant p < \infty. \end{cases}$$

Proof. We first prove the hint; WLOG assume $x > y$, so we can drop the cumbersome $|\cdot|$. If $p \in (0,1)$, then Problem 17(a) suggests that for $a, b > 0$ we have $(a+b)^p \leqslant a^p + b^p$. Setting $a = x - y$ and $b = y$, we obtain

$$x^p \leqslant (x-y)^p + y^p \implies x^p - y^p \leqslant (x-y)^p.$$

If $p \geqslant 1$, the MVT on the function $f(x) = x^p$ implies

$$x^p - y^p = (x-y)f'(\xi) = p(x-y)(\xi^{p-1}) \qquad \text{for some } \xi \in [x,y].$$

It is clear that $x^{p-1} \leqslant x^{p-1} + y^{p-1}$, so we are done proving the hint. END OF PROOF OF HINT.

(a) The inequality $\int ||f|^p - |g|^p| \, \mathrm{d}\mu \leqslant \int |f-g|^p \, \mathrm{d}\mu$ directly follows from the hint. It is also clear that Δ is non-degenerate and symmetric, and if $f, g \in L^p(\mu)$ then $\delta(f,g)$ is finite. Finally, Δ satisfies the triangle inequality, as (for $f, g, h \in L^p(\mu)$)

$$\Delta(f,h) - \Delta(f,g) = \int |f-h|^p - |f-g|^p \, \mathrm{d}\mu \leqslant \int |g-h|^p \, \mathrm{d}\mu = \Delta(g,h),$$

so $\Delta(f,h) \leqslant \Delta(f,g) + \Delta(g,h)$.

The proof showing that L^p defined as such is complete is similar to that of Problem 18(c). Let $\{f_n\}_{n\geqslant 1}$ be a Cauchy sequence (w.r.t.Δ). In particular, we can extract a Cauchy subsequence $\{f_{n_k}\}$ such that $\Delta(f_{n_{k+1}}, f_{n_k}) < 2^{-k}$. Let $E_k := \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geqslant 2^{-k}\}$. Then we can estimate $\mu(E_k)$ using

$$2^{-kp}\mu(E_k) \leqslant \int_{E_k} |f_{n_{k+1}}(x) - f_{n_k}(x)|^p \, \mathrm{d}\mu \leqslant \int |f_{n_{k+1}}(x) - f_{n_k}(x)|^p \, \mathrm{d}\mu < 2^{-k}$$

which gives

$$\mu(E_k) \leqslant 1/2^{k(1-p)}. \tag{1}$$

From this, we see that $\bigcup_{k=1}^{\infty} E_k$ also has a finite measure, so (similar to what we have done before),

$$\mu(E) := \mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k\Big) = \lim_{m\to\infty} \mu\Big(\bigcup_{k=m}^{\infty} E_k\Big) = 0.$$

For any $x \notin E$, we can find a sufficiently large n_K, after which $|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}$ is always satisfied. That is, outside the null set E, $\{f_{n_k}\}$ converges. Define f whose restriction to E^c is that limit (and 0 in E, for instance). We claim that $\{f_n\} \to f$ w.r.t. Δ. By Fatou's lemma,

$$\int |f|^p \, \mathrm{d}\mu = \int_{E^c} |f|^p \, \mathrm{d}\mu = \int_{E^c} \liminf_{k\to\infty} |f_{n_k}|^p \, \mathrm{d}\mu \leqslant \liminf_{k\to\infty} \int |f_{n_k}|^p \, \mathrm{d}\mu,$$

so $f \in L^p$, and most importantly, $\Delta(f_{n_k}, f) \to 0$: given $\epsilon > 0$, there exists a sufficiently large n_K such that $\Delta(f_{n_k}, f_{n_j}) < \epsilon$ whenever $n_k, n_j > n_K$. Fix any such n_k, and we have (by Fatou's lemma, again)

$$\begin{aligned}\int |f - f_{n_k}|^p \, \mathrm{d}\mu &= \int \lim_{n_j\to\infty} |f_{n_j} - f_{n_k}|^p \, \mathrm{d}\mu = \int \liminf_{n_i\to\infty} |f_{n_i} - f_{n_k}|^p \, \mathrm{d}\mu \\ &\leqslant \liminf_{n_i\to\infty} \int |f_{n_i} - f_{n_k}|^p \, \mathrm{d}\mu = \liminf_{n_i\to\infty} \Delta(f_{n_i}, f_{n_k}) < \epsilon.\end{aligned}$$

Therefore the Cauchy sequence $\{f_n\}$ admits a convergent subsequence and it itself must be convergent too. Hence $L^p(\mu)$ is complete w.r.t. the metric Δ.

(b) Notice that $||f| - |g|| \leqslant |f - g|$, so

$$\begin{aligned}
\int ||f|^p - |g|^p| \, d\mu &\leqslant p \int ||f| - |g||(|f|^{p-1} + |g|^{p-}) \, d\mu && \text{(Hint)} \\
&\leqslant p \int |f - g|(|f|^{p-1} + |g|^{p-1}) \, d\mu \\
&= p \int |f - g| \cdot |f|^{p-1} \, d\mu + p \int |f - g| \cdot |g|^{p-1} \, d\mu \\
&\leqslant p \left\{ \int |f - g|^p \, d\mu \right\}^{1/p} \left\{ \int (|f|^{p-1})^{p/(p-1)} \, d\mu \right\}^{(p-1)/p} \\
&\quad + p \left\{ \int |f - g|^p \, d\mu \right\}^{1/p} \left\{ \int (|g|^{p-1})^{p/(p-1)} \, d\mu \right\}^{(p-1)/p} && \text{(Hölder's)} \\
&= p\|f - g\|_p (\|f\|_p^{p-1} + \|g\|_p^{p-1}) \leqslant \|f - g\|_p \cdot 2pR^{p-1}. && \square
\end{aligned}$$

Problem 25

Suppose μ is a positive measure on X and $f : X \to (0, \infty)$ satisfies $\int_X f \, d\mu = 1$. Prove that for every $E \subset X$ with $0 < \mu(E) < \infty$,

$$\int_E (\log f) \, d\mu \leqslant \mu(E) \log \frac{1}{\mu(E)}$$

and that, when $0 < p < 1$,

$$\int_E f^p \, d\mu \leqslant \mu(E)^{1-p}.$$

Proof. (1) We scale μ by defining a new measure $\mu' := \mu/\mu(E)$. Then $\mu'(E) = 1$ and we can apply Jensen's inequality to $\log$, a concave function. Note that the negative of a concave function is convex, so Jensen's inequality is simply reversed for concave functions:

$$\varphi \text{ concave} \implies \varphi\left(\int_\Omega f \, d\mu \right) \geqslant \int_\Omega \varphi \circ f \, d\mu.$$

Then,

$$\frac{1}{\mu(E)} \int_E \log f \, d\mu = \int_E \log f \, d\mu' \leqslant \log\left(\int_E f \, d\mu' \right) = \log\left(\frac{1}{\mu(E)} \int_E f \, d\mu \right) \leqslant \log(1/\mu(E)).$$

(2) Consider the convex function $\varphi(x) := -x^p$ on $(0, \infty)$. Jensen's inequality gives

$$-\left\{ \int_E f \, d\mu' \right\}^p \leqslant \int_E f^p \, d\mu' = -\int_E f^p \, d\mu'.$$

Since $\int_E f \, d\mu' = \int_E f \, d\mu/\mu(E)$, we can rewrite the above inequality:

$$\begin{aligned}
\frac{1}{\mu(E)} \int_E f^p \, d\mu = \int_E f^p \, d\mu' &\leqslant \left\{ \int_E f \, d\mu' \right\}^p \\
&= \left\{ \frac{1}{\mu(E)} \int_E f \, d\mu \right\}^p = \mu(E)^{-p} \left\{ \int_E f \, d\mu \right\}^p \leqslant \mu(E)^{-p}.
\end{aligned}$$

Rearranging gives $\int_E f^p \, d\mu \leqslant \mu(E)^{1-p}$, and we are done.

$\square$

Problem 26

If f is a positive measurable function on $[0,1]$, then which is larger,

$$\int_0^1 f(x)\log f(x)\,\mathrm{d}x \qquad \text{or} \qquad \int_0^1 f(s)\,\mathrm{d}s \int_0^1 \log f(t)\,\mathrm{d}t?$$

Solution. The first one is larger. Notice that the function $\varphi(x) := x\log(x)$ is convex on $(0,\infty)$ and that $\log$ is concave. Having these facts, we now begin the estimate:

$$\begin{aligned}
\int_0^1 f(x)\log f(x)\,\mathrm{d}x &= \int_0^1 \varphi\circ f\,\mathrm{d}x \geqslant \varphi\Big(\int_0^1 f\,\mathrm{d}x\Big) && (\varphi \text{ is convex})\\
&= \int_0^1 f(s)\,\mathrm{d}s\cdot\log\Big(\int_0^1 f(t)\,\mathrm{d}t\Big)\\
&\geqslant \int_0^1 f(s)\,\mathrm{d}s\cdot\int_0^1 \log f(t)\,\mathrm{d}t. && (\log \text{ is concave})
\end{aligned}$$

□