

Rudin RCA Chapter 6 Partial Exercises

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Problem 1

If μ is a complex measure on a σ -algebra \mathfrak{M} and if $E \in \mathfrak{M}$, define

$$\lambda(E) := \sup \sum |\mu(E_i)|$$

where the supremum is taken on all *finite* partitions $\{E_i\}$ of E . Does it follow that $\lambda = |\mu|$?

Solution. Yes. It is clear that $\lambda \leq |\mu|$, and for the latter, let $\{\tilde{E}_i\}_{i=1}^n$ be an arbitrary *infinite* partition of E . Since μ is a complex measure, the series

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

converges and therefore converges absolutely because $\mu(E)$ does not change under permutations of $\{E_i\}$.

Let $\epsilon > 0$ be given. There exists a sufficiently large N satisfying

$$\sum_{i=N}^{\infty} |\mu(E_i)| < \epsilon.$$

We define \tilde{E} as the (disjoint) union of E_N, E_{N+1}, \dots . Then $\{E_1, \dots, E_{N-1}, \tilde{E}\}$ is a *finite* partition of E , and

$$\begin{aligned} \sum_{i=1}^{\infty} |\mu(E_i)| &= \sum_{i=1}^{N-1} |\mu(E_i)| + \sum_{i=N}^{\infty} |\mu(E_i)| < \sum_{i=1}^{N-1} |\mu(E_i)| + \epsilon \\ &\leq \underbrace{\sum_{i=1}^{N-1} |\mu(E_i)| + |\mu(\tilde{E})|}_{\text{a finite partition's sum}} + \epsilon \leq \lambda(E) + \epsilon. \end{aligned}$$

Since our original $\{E_i\}$ is arbitrary, the above inequality holds for any of them, and thus $|\mu|(E) = \sup \sum \leq \lambda(E) + \epsilon$, proving the other direction of the inequality. \square

Proof. Since $p \equiv 1 \pmod{5}$, we have either $p \equiv 1 \pmod{10}$ or $p \equiv 6 \pmod{10}$, but the latter implies p is even, and the only even prime 2 does not satisfy $2 \equiv 6 \pmod{10}$, so $(p \equiv 1 \pmod{5}) \Rightarrow (p \equiv 1 \pmod{10})$. \square

Problem 2

Prove that the example given at the end of Theorem 6.10 has the stated properties: if μ is the Lebesgue measure on $(0, 1)$ and λ is the counting measure on the σ -algebra of all Lebesgue measurable sets in $(0, 1)$, then λ has no Lebesgue decomposition relative to μ ; although \ll