Lemma

For every graph $G = (V, E)$, there exists a vertex v such that for all edge e incident on v, the set S of all nodes t such that e is on a shortest path from v to t satisfies $|S| \leqslant |V|/2$. Consequently, the game described in HW4 problem 4 terminates in at most $\log_2 n$ queries.

Proof. We define a total distance function $\varphi : V \to \mathbb{R}$ by

$$\varphi(v) = \sum_{u \in V} d(v, u) \qquad \text{where } d(v,u) \text{ is the shortest distance between } u \text{ and } v. \tag{1}$$

We claim $v_0 := \operatorname{argmin}_v \varphi(v)$ is the starting vertex we seek.
Let $e = (v_0, w)$ and S be defined as previously discussed. By construction we have

$$d(v_0, u) = d(v_0, w) + d(w, u) \qquad \text{for all } u \in S$$

and meanwhile by definition of shortest path between w and any u,

$$d(w, u) \leqslant d(v_0, w) + d(v_0, u) \implies d(v_0, u) \geqslant d(w, u) - d(v_0, w) \qquad \text{for all } u \notin S.$$

Therefore

$$\begin{aligned} \varphi(v_0) &= \sum_{u \in S} d(v_0, u) + \sum_{u \notin S} d(v_0, u) \\ &\geqslant \sum_{u \in S} [d(v_0, w) + d(w, u)] + \sum_{u \notin S} [d(w, u) - d(v_0, w)] \\ &= \varphi(w) + d(v_0, w)(|S| - |S^c|). \end{aligned} \tag{2}$$

Since v_0 minimizes φ we must have $|S| \leqslant |S^c|$, i.e., $|S| \leqslant |V|/2$.
The rest is easy — note that the proof would still hold if we replace (1) and (2) by

$$\varphi : V' \to \mathbb{R}, v \mapsto \sum_{u \in V'} d(v, u) \qquad \text{and} \qquad \varphi(v_0) = \sum_{u \in V' \cap S} d(v_0, u) + \sum_{u \in V' \cap S^c} d(v_0, u)$$

where $V' \subset V$ is arbitrary. In particular, if the j^{th} round of the game outputs a set S_j, we begin the $(j+1)^{\text{th}}$ iteration with parameters by setting the domain of our next φ as S_j. In the end, the number of possible candidates, i.e., $|S_j|$, becomes 1, and we obtain the only candidate for t in at most $\log_2 n$ queries. □