MATH 507a Homework 1

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Problem 1: Durrett 1.2.5

Suppose X has continuous density f, $\mathbb{P}(\alpha \leqslant X \leqslant \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = (y - b)/a$ so the answer is $(1/a)f((y - b)/a)$.

Proof. Inverse function theorem states that $(g^{-1})'(x) = 1/g'(g^{-1}(x))$. Let X has density f and distribution F, and let $g(X)$ has density h and distribution H. By assumption, H and h are differentiable. It follows that

$$h(y) = \frac{\mathrm{d}}{\mathrm{d}y}H(y) = \frac{\mathrm{d}}{\mathrm{d}y}F(g^{-1}(y)) = f(g^{-1}(y))/g'(g^{-1}(y)),$$

for y defined, namely $y \in (g(\alpha), g(\beta))$. $\square$

Problem 2: Durrett 1.2.6

Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$ the lognormal distribution.

Solution. Let f be the distribution function of X. Then

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

Using the previous question, let $g(X) := \exp(X)$, and we obtain

$$\begin{aligned} g(y) &= f(g^{-1}(y))/g'(g^{-1}(y)) = f(\log y)/g'(\log y) \\ &= f(\log y)/[\exp(\log y)] = y^{-1} f(\log y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-(\log y-\mu)^2/(2\sigma^2)}. \end{aligned}$$

Problem 3

Suppose $\mathcal{E}$, $\mathcal{F}$ are subsets of the power set $\mathcal{P}(X)$ with $\mathcal{E} \subset \mathcal{F} \subset \sigma(\mathcal{E})$. Show that $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$.

Proof. Since $\mathcal{E} \subset \mathcal{F}$ it it trivial that $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$. Conversely, since $\sigma(\mathcal{E})$ is the smallest σ-field containing $\mathcal{E}$ and $\sigma(\mathcal{F})$ is a field containing $\sigma(\mathcal{E})$ and thus $\mathcal{E}$, it must be contained in $\sigma(\mathcal{E})$. This completes the proof. $\square$

Problem 4

Suppose $A_1, \ldots, A_n$ are subsets of Ω and there are no relations among these sets. Show that $\sigma(\{A_1, \ldots, A_n\})$ consists of 2^{2^n} sets.

Proof. We partition Ω into 2^n sets $S_0, \ldots, S_{2^n-1}$, each set consisting of points belonging to precisely a certain subcollection of A_i's. For $0 \leqslant k \leqslant 2^n - 1$, we write k as $\sum_{j=0}^{n} c_j 2^j$ where $c_j \in$ where $c_j \in \{0, 1\}$ and define

$$S_k := \{x \in \Omega : x \in A_i \text{ iff } c_i = 1\} = \bigcap_{i:c_i=1} A_i \cap \bigcap_{j:c_j=0} A_j^c.$$

Since by assumption there are no relations among these sets, we indeed obtain a partition. It is clear that each S_i is a finite intersection of A_i's and their complements, so they appear in $\sigma(\{A_1, \ldots, A_n\})$, and so do the union of any of the S_i's, which provides 2^{2^n} elements. On the other hand, since each A_i can be represented as a disjoint union of the S_i's, the result of any set operations on A_i's can still be represented by set operations on the S_i's. That is, $|\sigma(\{A_i\})| \leqslant 2^{2^n}$. Combining the result we obtain $|\sigma(\{A_i\})| = 2^{2^n}$. $\square$

Problem 5

Suppose $\{A_n\}$ are almost disjoint: $\mathbb{P}(A_n \cap A_m) = 0$ for $n \geqslant m$. Show that $\mathbb{P}(\bigcup_{n\geqslant 1} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

Proof. Define $C_0 := \varnothing$ and $C_n := A_n \cap \bigcap_{i=1}^{n-1} A_i$ for $n \geqslant 2$. For $n \geqslant 1$, also define $B_n := A_n \backslash C_n$. By doing so, the B_n's are pairwise disjoint with $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$ for all n. Then,

$$\mathbb{P}(\bigcup_{n\geqslant 1} A_n) = \mathbb{P}(\bigcup_{n\geqslant 1} B_n \cup C_n) = \mathbb{P}(\bigcup_{n\geqslant 1} B_n) \cup \mathbb{P}(\bigcup_{n\geqslant 1} C_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) = \sum_{i=1}^{\infty} \mathbb{P}(A_n).$$ $\square$

Problem 6

(1) Let X have density f. Find the density of X^4.

(2) in particular, if X is standard normal, what is the density of X^4?

Solution. Using definition,

$$\begin{aligned}\mathbb{P}(X^4 \leqslant x) &= \mathbb{P}(X^2 \leqslant \sqrt{x}) = \mathbb{P}(-x^{1/4} \leqslant X \leqslant x^{1/4}) \\ &= F(x^{1/4}) - F(-x^{1/4})\end{aligned}$$

where F is the distribution function of X. Differentiating gives the density

$$\begin{aligned}f_{X^4}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x^{1/4}) - F(-x^{1/4})) \\ &= \frac{1}{4x^{3/4}} f(x^{1/4}) + \frac{1}{4x^{3/4}} f(-x^{1/4}).\end{aligned}$$

In particular if X is the standard normal then $f(x^{1/4}) = f(-x^{1/4})$ so

$$f_{X^4}(x) = \frac{1}{2x^{3/4}} f(x^{1/4}) = \frac{1}{2\sqrt{2\pi}x^{3/4}} \exp(-\sqrt{x}/2).$$

Problem 7

Let $\omega_1 \neq \omega_2$ in Ω and let $\mathcal{G}$ be a collection of subsets of Ω such that each $G \in \mathcal{G}$ contains either both w_1, w_2 or neither. Let $\mathcal{F} = \sigma(\mathcal{G})$. Show that for every random variable X defined on $(\Omega, \mathcal{F})$ we have $X(\omega_1) = X(\omega_2)$.

Proof. By the "general principle," since $\mathcal{G}$ generates $\sigma(\mathcal{G})$ and every set in $\mathcal{G}$ either contains both ω_1 and ω_2 or neither, any set in $\sigma(\mathcal{G})$ shares the same property. If a r.v. X on $(\Omega, \mathcal{F})$ has $X(\omega_1) \neq X(\omega_2)$ then $X^{-1}(X(\omega_1))$ contains ω_1 but not ω_2, so $X^{-1}(X(\omega_1)) \notin \mathcal{F}$, contradiction. Hence all r.v. X on $(\Omega, \mathcal{F})$ must satisfy $X(\omega_1) = X(\omega_2)$. □

Problem 8

Let $\{\mu_n\}$ be measures on $(\Omega, \mathcal{F})$ and $c_n > 0$ with $\sum_{n=1}^{\infty} c_n \mu_n(\Omega) = 1$. Show that $\sum_{n=1}^{\infty} c_n \mu_n$ is a probability measure.

Proof. For convenience denote the summed to-be measure ν. It is clear that $\nu : \mathcal{F} \to [0,1]$ with $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$. It remains to show countable additivity of disjoint sets. Let $\{E_i\}$ be pairwise disjoint. On one hand we have

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{n=1}^{\infty} c_n \mu_n\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} c_n \mu_n(E_i) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_n \mu_n(E_i) = \sum_{i=1}^{\infty} \nu(E_i).$$

The double sum is interchangeable because each $c_n \mu_n(E_i)$ is nonnegative and we can therefore apply (Fubini-)Tonelli w.r.t. the counting measure. □

Problem 9

We say a set $A \subset \{1, 2, \ldots\}$ has asymptotic density $\rho(A)$ if

$$\lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = \rho(A).$$

For every $A \subset \{1, 2, \ldots\}$ we define the block densities

$$\rho_k(A) = \frac{|A \cap (2^{k-1}, 2^k]|}{2^k - 2^{k-1}}, k \geqslant 0.$$

(1) Show that A has density $\rho(A) = \lambda$ if and only if $\rho_k(A) \to \lambda$ as $k \to \infty$.

(2) Let $\mathcal{F} = \{A \subset \{1, 2, \ldots\} : A \text{ has an asymptotic density}\}$. Show that ρ is finitely additive on $\mathcal{F}$.

(3) Prove or disprove $\mathcal{F}$ is a field.

Proof. (1) Note that for any number of form $2^n - 1$,

$$\frac{|A \cap \{1, 2, ..., 2^n - 1\}|}{n} \approx n^{-1} \sum_{j=1}^{n} |A \cap (2^{j-1}, 2^j]| = \sum_{k=1}^{n} 2^{k-n-1} \rho_k(A).$$

(Not = because the LHS's denominator contains 1 but the RHS does not, but $1/n \to 0$ as $n \to \infty$ and so this extra term is negligible.) If $\rho_k(A) \to \lambda$, then given $\epsilon > 0$, $|\rho_k(A) - \lambda| < \epsilon$ for sufficiently large k, say for $k \geqslant N$. Then for $n \geqslant N$,

$$\begin{aligned}\left|\lambda - \sum_{k=1}^{n} 2^{k-n-1} \rho_k(A)\right| &\leqslant \sum_{k=1}^{n} 2^{k-n-1} |\lambda - \rho_k(A)| + \sum_{j=k-n-2}^{-\infty} 2^j \lambda \\ &\leqslant \epsilon + 2^{k-n-1}\lambda \leqslant \epsilon + 2^{k-n-1}.\end{aligned}$$

Letting $n \to \infty$ we see

$$\lim_{n\to\infty} \left|\lambda - \sum_{k=1}^{n} 2^{k-n-1} \rho_k(A)\right| = 0,$$

namely $\rho(A) = \lambda$.

(2) This is immediate from (a). Suppose $A_1, ..., A_n$ have densities $\lambda_1, ..., \lambda_n$. By (a), we must have

$$\lim_{k\to\infty} \rho_k(A_i) = \lambda_i \qquad \text{for } 1 \leqslant i \leqslant n.$$

For each k, however, $A \cap (2^{k-1}, 2^k] = \bigcup_{i=1}^{n} A_i \cap (2^{k-1}, 2^k]$ because the A_i's are disjoint. Therefore

$$\lim_{k\to\infty} \rho_k(\bigcup_{i=1}^{n} A_i) = \lim_{k\to\infty} \sum_{i=1}^{n} \rho_k(A_i) = \sum_{i=1}^{n} \lim_{k\to\infty} \rho_k(A_i) = \sum_{i=1}^{n} \rho(A_i),$$

assuming the finite union is still in $\mathcal{F}$.

(3) No. Consider $A := \bigcup_{n\geqslant 1} (2^{n-1}, 2^n] = (1, 2] \cup (4, 8] \cup (16, 32] \cup ....$ Note that if k is even then $\rho_k(A) = 1$, and if k is odd then $\rho_k(A) = 0$. Now consider $\sum_{k=1}^{n} 2^{k-n-1} \rho_k(A)$. It is clear that

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \sum_{k=1}^{n} 2^{k-n-1} \rho_k(A) = \frac{1}{3} \text{ while } \lim_{\substack{n\to\infty \\ n \text{ odd}}} \sum_{k=1}^{n} 2^{k-n-1} \rho_k(A) = \frac{2}{3}.$$

That is, $\liminf_{n\to\infty} \frac{|A \cap \{1, ..., n\}|}{n} \leqslant \frac{1}{3} < \frac{2}{3} \leqslant \limsup_{n\to\infty} \frac{|A \cap \{1, ..., n\}}{n}|$, so $A \notin \mathcal{F}$, although all intervals of for $(2^{k-1}, 2^k]$ are (and have asymptotic density 0 for being finite sets). □