

MATH 507a Homework 3

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September 28, 2022

Problem: Durrett 2.1.11

If X, Y are independent, integer-valued problems, we have

$$\mathbb{P}(X + Y = n) = \sum_m \mathbb{P}(X = m)\mathbb{P}(Y = n - m).$$

Show that if X, Y are independent Poisson with parameters λ and μ then $X + Y$ is Poisson with parameter $(\lambda + \mu)$.

Proof. Let Z denote $X + Y$. Then

$$\begin{aligned} \mathbb{P}(Z = z) &= \sum_{j=0}^z \mathbb{P}(X = j)\mathbb{P}(Y = z - j) \\ &= \sum_{j=0}^z \frac{e^{-\lambda} \lambda^j}{j!} \cdot \frac{e^{-\mu} \mu^{z-j}}{(z-j)!} = \sum_{j=0}^z \frac{e^{-\lambda} e^{-\mu} \lambda^j \mu^{z-j}}{j!(z-j)!} \\ &= \sum_{j=0}^z \frac{z!}{j!(z-j)!} \frac{e^{-\lambda} e^{-\mu} \lambda^j \mu^{z-j}}{z!} = \frac{e^{-z}}{z!} \sum_{j=0}^z \binom{z}{j} \lambda^j \mu^{z-j} \\ &= \frac{e^{-z}}{z!} (\lambda + \mu)^z. \end{aligned}$$

□

Problem: Durrett 2.1.14

Let $X, Y \geq 0$ be independent with d.f. F, G . Find the d.f. of XY .

Solution. The probability $\mathbb{P}(XY \leq z) = 0$ if $z < 0$. Assuming $z \geq 0$:

$$\begin{aligned} \mathbb{P}(XY \leq z) &= \iint 1_{\{xy \leq z\}} dF(x)dG(y) \\ &= \iint [1_{\{y=0\}} + 1_{\{y \neq 0, x \leq z/y\}}] dF(x)dG(y) \\ &= \mathbb{P}(Y = 0) + \int_{\mathbb{R}} F(z/y) dG(y) = \mathbb{P}(Y = 0) + \int_{(0, \infty)} F(z/y) dG(y). \end{aligned}$$

Problem: Durrett 2.2.2

The L^2 weak law generalizes immediately to certain independent sequences. Suppose $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n X_m \leq r(n - m)$ for $m \leq n$ with $r(k) \rightarrow 0$ as $k \rightarrow \infty$. Show that $(X_1 + \dots + X_n)/n \rightarrow 0$ in probability.

Proof. Since $\mathbb{E}X_n = 0$ we have $\mathbb{E}S_n = 0$. Therefore $\mathbb{E}(S_n/n)^2 = \text{var}(S_n/n)$. Also, Cauchy-Schwarz gives

$$\mathbb{E}X_n X_m \leq \mathbb{E}|X_n X_m| \leq (\mathbb{E}X_n^2)^{1/2} (\mathbb{E}X_m^2)^{1/2} \leq r(0) < \infty.$$

Let $\epsilon > 0$ be given and let N be such that $|r(n)| < \epsilon$ for $n \geq N$. Then, for $n \geq N$,

$$\begin{aligned} \mathbb{E}(S_n/n)^2 &= \text{var}(S_n/n)^2 = n^{-2} \sum_{i,j} \mathbb{E}X_i X_j \\ &= n^{-2} \left(\sum_{i=1}^n \text{var}(X_i) + 2 \sum_{i < j} \text{cov}(X_i, X_j) \right) \\ &= n^{-2} \left(\sum_{i=1}^n \mathbb{E}X_i^2 + 2 \sum_{i < j} \mathbb{E}X_i X_j \right) \\ &\leq n^{-2} (nr(0) + 2 \sum_{i < j} r(j - i)) \\ &= \frac{r(0)}{n} + \frac{2}{n^2} [(n-1)r(1) + (n-2)r(2) + \dots + r(n-1)] \\ &\leq \frac{r(0)}{n} + \frac{2n \sum_{i=1}^n r(n-i)}{n^2} \\ &\leq \frac{r(0)}{n} + \frac{2}{n} \left[\sum_{i=1}^N r(N-i) + \sum_{i=N}^n r(i) \right] \\ &\leq \frac{r(0)}{n} + \frac{2C}{n} + \frac{2(n-N)\epsilon}{n}. \end{aligned}$$

The first two terms converge to 0 as $n \rightarrow \infty$, and the last term converges to 2ϵ . Since ϵ is arbitrary we see $\mathbb{E}(S_n/n)^2 \rightarrow 0$. Since convergence in L^p implies convergence in probability, we are done. \square

Problem: Durrett 2.2.6

Show that if $X \geq 0$ is integer valued then $\mathbb{E}X = \sum_{n \geq 1} \mathbb{P}(X \geq n)$. Find a similar expression for $\mathbb{E}X^2$.

Proof. For the first one: $X = \sum_{n=1}^{\infty} 1_{\{X \geq n\}}$ so

$$\mathbb{E}X = \mathbb{E} \sum_{n=1}^{\infty} 1_{\{X \geq n\}} = \sum_{n=1}^{\infty} \mathbb{E}1_{X \geq n} = \sum_{n \geq 1} \mathbb{P}(X_n).$$

For the second part,

$$\mathbb{E}X^2 = \sum_{x=1}^{\infty} x^2 \mathbb{P}(X = x) = \sum_{n=1}^{\infty} \sum_{m=1}^n m \mathbb{P}(X = n) = \sum_{n \geq 1} (2n-1) \mathbb{P}(X \geq n).$$

In both parts, interchange of limits are allowed b/c the terms are nonnegative. \square

Problem 1

Let X_1, X_2, \dots be i.i.d. with

$$\mathbb{P}(X_1 = (-1)^k k) = \frac{c}{k^2 \log k}, \quad k \geq 2$$

where c is chosen so that the probabilities sum to 1.

- (1) Show that $\mathbb{E}|X_1| = \infty$ but there exists a constant A with $S_n/n \rightarrow A$ in probability.
- (2) Show that (a) becomes false if $(-1)^k$ becomes just k .

Proof. (1)

$$\mathbb{E}|X_1| = \sum_{k \geq 2} \frac{kc}{k^2 \log k} = c \sum_{k \geq 2} \frac{1}{k \log k} = \infty$$

by integral test and divergence of $\int_1^\infty \frac{1}{x \log x} dx$.

By WLLN, it suffices to show that $n\mathbb{P}(|X_1| \geq n) \rightarrow 0$ and that $A_n = \mathbb{E}[X_1 1_{|X_1| \leq n}]$ converges.

For the first claim,

$$\begin{aligned} n\mathbb{P}(|X_1| \geq n) &= n \sum_{k=n}^\infty \frac{c}{k^2 \log k} \leq \frac{cn}{\log n} \sum_{k=n}^\infty \frac{1}{k^2} \\ &\leq \frac{cn}{\log n} \int_{n-1}^\infty \frac{1}{x^2} dx = \frac{cn}{\log n} \cdot \frac{1}{n-1} \sim \frac{c}{\log n} \rightarrow 0. \end{aligned}$$

To show the second claim:

$$A_n = \sum_{k=2}^n (-1)^k k \mathbb{P}(X_1 = (-1)^k) = \sum_{k=2}^n \frac{(-1)^k ck}{k^2 \log k} = \sum_{k=2}^n (-1)^k \frac{c}{k \log k},$$

a series of alternating numbers with decreasing magnitude. Therefore A_n converges, and the limit is the A we seek by WLLN.

- (2) Without the alternating series, the series $\sum 1/(k \log k)$ diverges. By WLLN S_n/n would then converge to $\lim A_n = \infty$ so no such constant A exists.

□

Problem 2

Suppose we have n boxes and we distribute r balls among them. Each of the balls is equally likely to go into any box, independently of other balls. We consider a limit in which $r/n \rightarrow \lambda$ for some $0 \leq \lambda < \infty$. Let N_n be the number of empty boxes.

- (1) Express $N + n$ as a sum of indicators.
- (2) Find $\lim_n \mathbb{E}N_n/n$ expressed in terms of λ .
- (3) Show that $\text{var}(N_n/n) \rightarrow 0$.
- (4) What can you conclude about N_n/n as $n \rightarrow \infty$?

Proof. (1) $N_n = \sum 1_{A_i}$ where $A_i := \{\text{the } i^{\text{th}} \text{ box is empty}\}$.

(2) For each box, the probability that it remains empty after we put 1 ball is $1 - 1/n$ so the probability of not getting a ball after r have been put is $(1 - 1/n)^r$. Hence $\mathbb{E}1_{A_i} = (1 - 1/n)^r$ and

$$\mathbb{E}(N_n/n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}1_{A_i} = (1 - 1/n)^r = (1 - 1/n)^{1/n \cdot (r/n)}.$$

Taking limits we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}(N_n/n) = e^{-\lambda}.$$

(3) Note that $\mathbb{E}1_{A_i \cap A_j} = (1 - 2/n)^r$, i.e., the probability of two chosen boxes both being empty, and that

$$\mathbb{E}N_n^2 = \mathbb{E}\left(\sum_{i=1}^n 1_{A_i}\right)^2 = \mathbb{E}\sum_{i=1}^n 1_{A_i} + \mathbb{E}\sum_{i \neq j} 1_{A_i \cap A_j} = n(1 - 1/n)^r + n(n-1)(1 - 2/n)^r.$$

Therefore,

$$\begin{aligned} \text{var}(N_n/n) &= n^{-2}(\mathbb{E}N_n^2 - (\mathbb{E}N_n)^2) \\ &= n^{-2}\left(n\left(1 - \frac{1}{n}\right)^r + n(n-1)\left(1 - \frac{2}{n}\right)^r - n^2\left(1 - \frac{1}{n}\right)^{2r}\right) \\ &= n^{-1}\left(1 - \frac{1}{n}\right)^r + \frac{n-1}{n}\left(1 - \frac{2}{n}\right)^r - \left(1 - \frac{1}{n}\right)^{2r}. \end{aligned}$$

Taking limits gives

$$\text{var}(N_n/n) = \frac{e^{-\lambda}}{n} + e^{-2\lambda} - e^{-2\lambda} \rightarrow 0.$$

(4) By Markov,

$$\mathbb{P}\left(\left|\frac{N_n}{n} - e^{-\lambda}\right| > \epsilon\right) \leq \frac{\text{var}(N_n/n)}{\epsilon^2} \rightarrow 0 \quad \text{for all } \epsilon > 0,$$

so N_n/n converges to $e^{-\lambda}$ in probability. □

Problem 3

Suppose A_1, A_2, \dots are events with $\mathbb{P}(A_n) \rightarrow 0$ and $\sum_{n \geq 1} \mathbb{P}(A_n \cap A_{n+1}^c) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.

Proof. By B-C we know $\mathbb{P}(A) := \mathbb{P}(\{x : x \in A_n \cap A_{n+1} \text{ i.o.}\}) = 0$. For convenience denote $B_n := \{A_n \cap A_{n+1}^c \text{ i.o.}\}$. If $x \in B$ then in particular $x \in A_n \text{ i.o.}$ so we have $B \subset A$. Let $C = B \setminus A$. If $x \in C$, then $x \in A_n \cap A_{n+1}^c$ only finitely often, so $x_n \in A_n$ for sufficiently large $n \geq N$. In particular $x \in A$. Therefore $x \in A_n \text{ i.o.}$, contradiction. Therefore $C = \emptyset$ and $B = A$, so we are done. □

Problem 4

Show that for any sequence of events A_n ,

$$\mathbb{P}(\limsup_n A_n) \geq \limsup_n \mathbb{P}(A_n) \quad \text{and} \quad \mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n).$$

Proof. We only show the second; once this is done, take complement and we automatically get the first.

$$\mathbb{P}(\liminf_n A_n) = \int \liminf_n 1_{A_n} d\mathbb{P} \leq \liminf_n \int 1_{A_n} d\mathbb{P} = \liminf_n \mathbb{P}(A_n). \quad \square$$

Problem 5

Let X, Y be r.v.'s with $\mathbb{E}X^2 < \infty, \mathbb{E}Y^2 < \infty$.

- (1) For what a, b is $\mathbb{E}([Y - (aX + b)]^2)$ minimized? Express a, b in terms of $\mathbb{E}X, \mathbb{E}Y, \sigma_X, \sigma_Y$.
- (2) Show that for the optimal a, b from part (a), the squared error as a fraction of σ_Y^2 is

$$\frac{\mathbb{E}[Y - (aX + b)]^2}{\sigma_Y^2} = 1 - \rho(X, Y)^2.$$

- (3) Show that for optimal a, b from part (a), given $\epsilon, \eta > 0$ there exists $\delta > 0$ such that

$$|\rho(X, Y)| > 1 - \delta \implies \mathbb{P}(|Y - (aX + b)| > \epsilon) < \eta.$$

This says for $\rho(X, Y)$ near ± 1 , Y must lie near $y = ax + nb$ with high probability.

Proof. (1) Since

$$\begin{aligned} \mathbb{E}[Y - (aX + b)]^2 &= \mathbb{E}Y^2 - 2a\mathbb{E}XY - 2\mathbb{E}Y(aX + b) + (aX + b)^2 \\ &= \mathbb{E}Y^2 - 2a\mathbb{E}XY - 2b\mathbb{E}Y + a^2\mathbb{E}X^2 + 2ab\mathbb{E}X + b^2 \end{aligned}$$

setting the gradients to 0, we have

$$-2\mathbb{E}XY + 2a\mathbb{E}X^2 + 2b\mathbb{E}X = 0$$

and

$$-2\mathbb{E}Y + 2a\mathbb{E}X + 2b = 0.$$

Solving the system gives

$$\begin{cases} -2\mathbb{E}XY + 2a\mathbb{E}X^2 + 2b\mathbb{E}X = 0 \\ -2\mathbb{E}X\mathbb{E}Y + 2a(\mathbb{E}X)^2 + 2b\mathbb{E}X = 0 \end{cases} \implies 2a[\mathbb{E}X^2 - (\mathbb{E}X)^2] + 2\mathbb{E}X\mathbb{E}Y - 2\mathbb{E}XY = 0,$$

so $a = (\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y)/(\mathbb{E}X^2 - (\mathbb{E}X)^2) = \text{cov}(X, Y)/\sigma_X^2$. Plugging this in, we obtain

$$b = \mathbb{E}Y - a\mathbb{E}X = \mathbb{E}Y - \frac{\text{cov}(X, Y)}{\sigma_X^2}\mathbb{E}X.$$

Finally, since $[Y - (aX + b)]^2$ is convex, the critical point must be the global minimum.

(2)

$$\begin{aligned}
\mathbb{E}[(Y - (aX + b))]^2 &= \text{var}(Y - (aX + b)) + (\mathbb{E}[Y - (aX + b)])^2 \\
&= \text{var}(Y - aX + b) + (\mathbb{E}Y - a\mathbb{E}X - b)^2 \\
&= \text{var}(Y - aX + b) = \text{var}(Y) + \text{var}(aX + b) - 2\text{cov}(Y, aX + b) \\
&= \text{var}(Y) + a^2 \text{var}(X) - 2a \text{cov}(X, Y) \\
&= \text{var}(Y) + \frac{\text{cov}^2(X, Y)}{\sigma_X^2} - \frac{2\text{cov}(X, Y)\text{cov}(X, Y)}{\sigma_X^2} \\
&= \text{var}(Y) - \frac{\text{cov}^2(X, Y)}{\sigma_X^2}.
\end{aligned}$$

Now dividing both sides by σ_Y^2 , we obtain

$$\frac{\mathbb{E}([Y - (aX + b)])^2}{\sigma_Y^2} = 1 - \rho(X, Y)^2.$$

(3) Let ϵ, η be given. From above, and by Chebyshev's inequality, we have

$$\mathbb{P}(|Y - (aX + b)| > \epsilon) \leq \frac{\text{var}(Y - (aX + b))}{\epsilon^2} = \frac{\mathbb{E}(Y - (aX + b))^2}{\epsilon^2} = \frac{\sigma_Y^2(1 - \rho(X, Y)^2)}{\epsilon^2}.$$

Therefore, it suffices to ensure

$$\frac{\sigma_Y^2(1 - \rho(X, Y)^2)}{\epsilon^2} < \eta \iff 1 - \rho(X, Y)^2 < \frac{\epsilon^2 \eta}{\sigma_Y^2} \iff \rho(X, Y) > \sqrt{1 - \epsilon^2 \eta / \sigma_Y^2}. \quad \square$$