MATH 507a Homework 4

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Problem: D2.3.6

Show (a) that $d(X,Y) = \mathbb{E}|X-Y|/(1+|X-Y|)$ defines a metric on the set of random variables and (b) that $d(X_n, X) \to 0$ iff $X_n \to X$ in probability.

Proof. It is clear that $d(X,Y) \geqslant 0$. If $d(X,Y) = 0$, then $|X-Y| = 0$ a.s., and $d(X,Y) = d(Y,X)$. For triangle inequality,

$$\frac{|X-Z|}{1+|X-Z|} \leqslant \frac{|X-Y|+|Y-Z|}{1+|X-Y|+|Y-Z|} \leqslant \frac{|X-Y|}{1+|X-Y|} + \frac{|Y-Z|}{1+|Y-Z|}$$

where the first inequality is because $x \mapsto x/(1+x)$ is increasing and $|X-Z| \leqslant |X-Y|+|Y-Z|$, and the second because

$$\frac{a+b}{1+a+b} \leqslant \frac{a}{1+a} + \frac{b}{1+b}$$

for $a, b \geqslant 0$, or by the hint. Taking expected value, we see d is a metric.

For (b), let $\epsilon > 0$. If $d(X_n, X) \to 0$, then

$$\mathbb{P}(|X_n - X| > \epsilon) \leqslant \frac{1+\epsilon}{\epsilon}\mathbb{E}\left(\frac{|X_n - X|}{1+|X_n - X|}\right) \leqslant \frac{1+\epsilon}{\epsilon} d(X_n, X) \to 0.$$

Conversely assume $X_n \to X$ in probability. Then,

$$\begin{aligned} d(X_n, X) &= \mathbb{E}\left(\frac{|X_n - X|}{1+|X_n - X|}\right) \\ &= \mathbb{E}\left(\frac{|X_n - X| 1_{\{|X_n - X| > \epsilon\}}}{1+|X_n - X|} + \frac{|X_n - X| 1_{\{|X_n - X| \leqslant \epsilon\}}}{1+|X_n - X|}\right) \\ &\leqslant \mathbb{P}(|X_n - X| > \epsilon) + \frac{\epsilon}{1+\epsilon}. \end{aligned}$$

Again, the last inequality makes use of the fact that $x \mapsto x/(1+x)$ is increasing. Since ϵ is arbitrary, we obtain $d(X_n, X) \to 0$ as $n \to \infty$. □

Problem: D2.3.13

If X_n is any sequence of random varaibles, there are constants $c_n \to \infty$ such that $X_n / c_n \to 0$ a.s.

Proof. Using Borel-Cantelli, if c_n are such that $\mathbb{P}(|X_n/c_n| > 1/n) < 2^{-n}$, then $\sum_{n=1}^{\infty} \mathbb{P}(|X_n/c_n| > 1/n) < \infty$, so $\mathbb{P}(|X_n/c_n| > 1/n \text{ i.o.}) = 0$, so $\mathbb{P}(|X_n/c_n| \leqslant 1/n \text{ eventually}) = 1$. We are almost done! Let $\epsilon > 0$ be given. There exists n_0 sufficiency large with $1/n_0 < \epsilon$. But then $\mathbb{P}(|X_n/c_n| \leqslant 1/n_0 \text{ eventually}) = 1$. Now letting $\epsilon \to 0$, and we see $X_n \to X$ a.s. □

Problem: 2.3.15(ii)

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i > x) = e^{-x}$ and let $M_n = \max_{m \leqslant n} X_m$. Show that $M_n/\log n \to 1$ a.s.

Proof. Let y_n be any sequence. Clearly, if $y_n/\log n$ i.o., we have $\max_{m\leqslant n} y_m/n > c$ i.o. Conversely, let $k(n)$ be a subsequence of indices such that $\max_{m\leqslant k(n)} y_m/\log n > c$ i.o. Since log is increasing, if $\max_{m\leqslant k(n)} y_n/\log n > c$, so does $\operatorname{argmax}_{m\leqslant k(n)} y_n/\log m$, so we indeed obtain a sequence of form $y_n/\log n$ that is greater than c. Hence $y_n/\log n$ i.o.

We first show $\liminf M_n/\log n \geqslant 1$. Let $\epsilon > 0$ and it suffices to show $\mathbb{P}(M_n/\log n < 1-\epsilon \text{ i.o.}) = 0$. This is because

$$\mathbb{P}(M_n/\log n < 1-\epsilon) = \prod_{i=1}^{n} \mathbb{P}(X_i \leqslant (1-\epsilon)\log n) = (1-n^{\epsilon-1})^n \leqslant e^{-n^{\epsilon}},$$

so

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n/\log n < 1-\epsilon) \leqslant \sum_{n=1}^{\infty} e^{-n^{\epsilon}}.$$

Since this is summable, we have $\mathbb{P}(M_n/\log n < 1-\epsilon \text{ i.o.}) = 0$ by Borel-Cantelli, i.e., $\liminf m_n/\log n \geqslant 1$.

Conversely, using the hint and $\limsup X_n/\log n = 1$, we see that $\limsup M_n/\log n$ cannot be greater than 1, for if $M_n/\log n > 1+\epsilon$ i.o. for some ϵ then so does $X_n/\log n$, and this contradicts $\limsup X_n/\log n = 1$. Therefore, $M_n/\log n \to 1$ a.s. □

Problem: D2.4.1

Suppose the i^{th} light bulb burns for an amount of time X_i then remains burned out for time Y_i before being replaced. Suppose X_i, Y_i are positive and independent with X's having distribution F and Y's having distribution G, both of which have finite mean. Let R_t be the amount of time in $[0,t]$ that we have a workling light bulb. Show that $R_t/t \to \mathbb{E}X_i/(\mathbb{E}X_i + \mathbb{E}Y_i)$ almost surely.

Proof. Similar to the normal janitor problem, let $N_t = \inf\{n : X_1 + Y_1 + \ldots + X_n + Y_n \leqslant t\}$. Then

$$\frac{X_1 + \ldots + X_{N_t}}{t} \leqslant \frac{R_t}{t} \leqslant 1 - \frac{Y_1 + \ldots + Y_{N_t}}{t}.$$

Taking limits using SLLN, $\sum X_i/N_t \to \mathbb{E}X_1, \sum Y_i/N_t \to \mathbb{E}Y_1$, and $N_t/t \to 1/(\mathbb{E}X_1 + \mathbb{E}Y_1)$, all of which are a.s. Then

$$\frac{R_t}{t} \to \frac{\sum X_i}{N_t}\frac{N_t}{t} = \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1} \text{ a.s.}$$

□

Problem 1

(1) Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = n^2 - 1) = 1/n^2$, $\mathbb{P}(X_n = -1) = 1 - 1/n^2$. Show that $\mathbb{E}X_n = 0$ for all n but $S_n/n \to 1$ a.s.

(2) Suppose that in (a) we replace n^2 and n^{-2} throughout by n and n^{-1}. Show that there are no constants μ_n such that $S_n/n - \mu_n \to 0$ a.s.

Proof. (1) It is clear that $\mathbb{E}X_n = 0$. Also, $\sum_{i=1}^{n} \mathbb{P}(X_i \neq -1) < \infty$ so by Borel-Cantelli, $\mathbb{P}(X_i \neq -1 \text{ i.o.}) = 0$. But if $X_i = -1$ eventually, $S_n/n \to -1$. Therefore $S_n/n \to -1$ a.s.

(2) Using Borel-Cantelli twice, since $\sum \mathbb{P}(X_n = n) = \sum \mathbb{P}(X_n = -1) = \infty$, we have $\mathbb{P}(X_n = n \text{ i.o.}) = \mathbb{P}(X_n = -1 \text{ i.o.}) = 1$. But then $\mathbb{P}(X_n/n = 1 \text{ i.o.}) = 1$ and $\mathbb{P}(X_n \to 0 \text{ in some subsequence}) = 1$. These together makes it impossible to find μ with $S_n/n - \mu \to 0$ a.s.

□

Problem 2

Show that $X_n \to X$ a.s. iff $\mathbb{P}(|X_n - X| > \epsilon \text{ i.o.}) = 0$ for all $\epsilon > 0$.

Prove the following variant: $X_n \to X$ a.s. iff there exists $\epsilon_n \to 0$ such that $\mathbb{P}(|X_n - X| > \epsilon_n \text{ i.o.}) = 0$.

Proof. (1) If $\mathbb{P}(|X_n - X| > \epsilon \text{ i.o.}) \neq 0$ for some ϵ, then X_n fails to converge to X on $\{|X_n - X| > \epsilon \text{ i.o.}\}$ and therefore the convergence is not a.s. Conversely, all events of form $A_k = \{|X_n - X| \leqslant 1/k \text{ for all large } n\}$ has $\mathbb{P}(A_k) = 1$, and so does their intersection A. On this set $X_n \to X$, so $X_n \to X$ a.s.

(2) The $\Longleftarrow$ is already shown in the previous part. For $\Longrightarrow$, since $X_n \to X$ a.s., for each k, there exists a sufficiently large N_k with

$$\mathbb{P}(\sup_{n \geqslant N_k} |X_n - X| \geqslant 1/k) < 1/k^2.$$

Summing over all such sets, the RHS is bounded, so by Borel-Cantelli

$$\mathbb{P}(\sup |X_n - X| \geqslant 1/k \text{ i.o.}) = 0.$$

Now let $\epsilon_n = 1/n$, and we are done.

□

Problem 3

(1) Let $U_1, U_2, \ldots$ be i.i.d. with values in (a, b) for some $0 < a < b < \infty$. Show that the variables $X_n = \prod_{i=1}^{n} U_i$ have a well-defined nonrandom exponential growth/decay rate $c \in \mathbb{R}$ in the sense that for all $\epsilon > 0$,

$$e^{(c-\epsilon)n} \leqslant X_n \leqslant e^{(c+\epsilon)n} \text{ for sufficiently large } n.$$

(2) Does such a c exist if U_1 is uniform in $[0, 2]$? If yes, find it; if no, is the growth/decay faster or slower

than exponential?

(3) Same as (b), if U_1 has distribution function $1/(1+\log(1/x))$ in $[0,1]$.

Proof. (1) Taking $\log$, it suffices to show that for some c,

$$(c-\epsilon)n \leqslant \sum_{i=1}^{n} U_i \leqslant (c+\epsilon)n \iff c-\epsilon \leqslant \frac{1}{n}\sum_{i=1}^{n}\log U_i \leqslant c+\epsilon.$$

Since U has values in (a,b) with $0<a$, $\mathbb{E}|\log U_i| < \log b < \infty$, so by SLLN, picking $c = \mathbb{E}\log U_i$ suffices.

(2) Since $\mathbb{E}\log U_i = \int_0^2 \log t/2 \,\mathrm{d}t = \log 2 - 1$, by the previous part, the exponential growth rate is just $\log 2 - 1$.

(3) In this part, the exponential growth rate does not exist because $\mathbb{E}\log U_i$ is not finite.

$$\begin{aligned}\mathbb{E}\log U_i &= \int_0^1 \log x \frac{\mathrm{d}}{\mathrm{d}x}F(x)\,\mathrm{d}x \\ &= \int_0^1 \frac{\log x}{x(1+\log(1/x))^2}\,\mathrm{d}x \\ &= -\frac{\log x}{\log x - 1}\bigg|_0^1 + \int_0^1 \frac{1}{x(\log x - 1)}\,\mathrm{d}x \\ &= -\frac{\log x}{\log x - 1}\bigg|_0^1 + \log(\log x - 1)\Big|_0^1 = -\infty.\end{aligned}$$

□

Problem 4

Let $\xi_1, \xi_2, \ldots$ be i.i.d. stanrard normal distributions. Show that

$$\limsup_{n\to\infty} \frac{\xi_n}{(2\log n)^{1/2}} = 1 \text{ a.s.}$$

Proof. For $\epsilon > 0$,

$$\begin{aligned}\sum \mathbb{P}\left(\frac{\xi_n}{\sqrt{2\log n}} \geqslant \sqrt{1+\epsilon}\right) &= \sum \mathbb{P}\left(\xi_n \geqslant \sqrt{2(1+\epsilon)\log n}\right) \\ &\leqslant \sum (2(1+\epsilon)\log n)^{-1/2}\exp(-2(1+\epsilon)\log n/2) \\ &= \sum \frac{n^{-(1+\epsilon)}}{\sqrt{2(1+\epsilon)\log n}}.\end{aligned}$$

Discarding finitely many terms to ensure $\log n > 1$, we can further bound the above by $\sum n^{-(1+\epsilon)}$ up to some constant scalar. Therefore, Borel-Cantelli implies

$$\mathbb{P}\left(\frac{\xi_n}{\sqrt{2\log n}} \geqslant \sqrt{1+\epsilon} \text{ i.o.}\right) = 0.$$

Conversely,

$$\begin{aligned}\sum \mathbb{P}\left(\frac{\xi_n}{(2\log n)^{1/2}} > \sqrt{1-\epsilon}\right) &= \sum \mathbb{P}(\xi_n > \sqrt{2(1-\epsilon)\log n}) \\ &\geqslant \sum \frac{1}{\sqrt{2(1-\epsilon)}\sqrt{\log n}}\left(1 - \frac{1}{2(1-\epsilon)\log n}\right)\exp(-2(1-\epsilon)\log n/2) \\ &= \sum \frac{1}{2\sqrt{(1-\epsilon)}\sqrt{\log n}}\left(1 - \frac{1}{2(1-\epsilon)\log n}\right) n^{-(1-\epsilon)}.\end{aligned}$$

Assuming we start with $0 < \epsilon < 0.5$, for all but finitely many terms,

$$1 - \frac{1}{2(1-\epsilon)\log n} < 0.5,$$

so we can further bound above by

$$\sum C \cdot n^{-(1-\epsilon)}(\log n)^{-1/2} \geqslant \sum C \cdot \frac{1}{n\log n} = \infty.$$

Therefore, by Borel-Cantelli, again,

$$\mathbb{P}\left(\frac{\xi_n}{\sqrt{2\log n}} \geqslant \sqrt{1-\epsilon}\ \text{i.o.}\right) = 1.$$

Combining these two identities we obtain the desired result. □