MATH 507a Homework 5

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Problem: D2.5.2

Let $p > 0$. Show that if $S_n/n^{1/p} \to 0$ a.s. then $\mathbb{E}|X_1|^p < \infty$.

Proof. If $S_n/n^{1/p} \to 0$ almost surely, then

$$\frac{X_n}{n^{1/p}} = \frac{S_n}{n^{1/p}} - \frac{S_{n-1}}{(n-1)^{1/p}} \frac{(n-1)^{1/p}}{n^{1/p}} \to 0$$

almost surely as well.
If $\mathbb{E}|X_1|^p = \infty$ then $\sum_{n=1}^{\infty} \mathbb{P}(|X_1| \geqslant n^{1/p}) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geqslant n^{1/p}) = \infty$, so by independence and second Borel-Cantelli, $|X_n| \geqslant n^{1/p}$ i.o., but this contradicts $X_n/n^{1/p} \to 0$ a.s., so $\mathbb{E}|X_1|^p < \infty$. □

Problem: D2.5.8

Let $X_1, X_2, \ldots$ be i.i.d. and not $\equiv 0$. SHow the radius of convergence of the power series $\sum_{n\geqslant 1} X_n(\omega)z^n$ is 1 a.s. or 0 a.s., according as $\mathbb{E}\log^+|X_1| < \infty$ or $= \infty$.

Proof. Recall the radius of convergence is given by $(\limsup_{n\to\infty}|X_n|^{1/n})^{-1}$.
If $\mathbb{E}\log^+|X_1| < \infty$ then

$$\sum_{n=1}^{\infty} \mathbb{P}(\log|X_n| > n) \leqslant \int_0^{\infty} \mathbb{P}(\log^+|X_1| > t)\, \mathrm{d}t < \infty$$

But then for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{1/n} > e^{\epsilon}) = \sum_{n=1}^{\infty} \mathbb{P}(\log^+|X_n| > n\epsilon) = \mathbb{E}\log^+|X_n|/\epsilon < \infty,$$

so $\mathbb{P}(|X_n|^{1/n} > 1 \text{ i.o.}) = 0$, which means $\limsup_{n\to\infty}|X_n|^{1/n} \leqslant 1$, and the radius of convergence $\geqslant 1$. On the other hand, if $|z| > 1$ then $\mathbb{P}(|X_n| > |z|^{-n}) = \mathbb{P}(|X_1| > |z|^{-n}) \to \mathbb{P}(|X_1| \neq 0) > 0$ as $n \to \infty$, so the series diverges with a nonzero probability. By Kolmogorov's $0-1$ law this implies the series diverges with probability 1. Therefore, the radius of convergence $\leqslant 1$, and combining both parts, it is 1.

If $\mathbb{E}\log^+|X_1| = \infty$ then

$$\sum_{n=1}^{\infty} \mathbb{P}(\log^+|X_n| > n) = \infty,$$

so for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{1/n} > e^{\epsilon}) = \sum_{n=1}^{\infty} \mathbb{P}(\log^+|X_n| > n\epsilon) = \infty.$$

By independence and second Borel-Cantelli, $|X_n| > e^{n\epsilon}$ i.o. Since ϵ is arbitrary, letting n be sufficiently large and we see $\limsup_{n\to\infty}|X_n|^{1/n} = \infty$ a.s. and the radius of convergence is 0 a.s. □

Problem: D2.5.10

Use Ottaviani's inequality to prove that if $X_1, X_2, \ldots$ are independent, then if $\lim S_n$ exists in probability, it also exists a.s.

Proof. If $S_n(\omega)$ fails to converge, then there exists $\epsilon > 0$ such that

$$\sup_{k>j\geqslant m} |S_{k,j}(\omega)| \geqslant \epsilon.$$

Since S_n converges in probability, given ϵ, δ, there exists N sufficiently large such that if $m \geqslant N$,

$$\sup_{k,j\geqslant m} \mathbb{P}(|S_{k,j}| \geqslant \epsilon) \leqslant \delta.$$

Using Ottaviani's inequality, along with the assumption $m \geqslant N$,

$$\mathbb{P}\left(\max_{m<j\leqslant n} |S_{m,j}| > 2\epsilon\right) \min_{m<k\leqslant n} \mathbb{P}(|S_{k,n}| \leqslant \epsilon) \leqslant \mathbb{P}(|S_{m,n}| > \epsilon),$$

where the second term bounded by $(1-\delta)$ from below, the third by δ from above, so

$$\mathbb{P}\left(\max_{m<j\leqslant n} |S_{m,j}| > 2\epsilon\right) \leqslant \frac{\delta}{1-\delta}.$$

Letting $n \to \infty, m \to \infty$, and $\epsilon, \delta \to 0$, we obtain almost sure convergence of $\sup|S_{m,j}|$ to 0, or equivalently, a.s. convergence of S_n. □

Problem: D2.7.3

Let $X_1, X_2, \ldots$ be i.i.d. Poisson with mean 1. Find $\lim_{n\to\infty}(1/n)\log\mathbb{P}(S_n \geqslant na)$ for $a > 1$.

Solution. The MGF $\varphi(\theta)$ is given by

$$\mathbb{E}\exp(tX) = \sum e^{\theta k}\frac{1}{ek!} = e^{-1}\sum\frac{(e^\theta)^k}{k!} = e^{-1}\exp(e^\theta) = \exp(e^\theta - 1),$$

so $\kappa(\theta) = e^\theta - 1$ and $\kappa'(\theta) = e^\theta$, giving $\theta_a = \log a$. Finally,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(S_n \geqslant na) = \gamma(a) = -a\theta_a + \kappa(\theta_a) = a - 1 - a\log a.$$

Problem 1

Let $A_1, A_2, \ldots$ be independent with $\mathbb{P}(A_k) = 1/k$ and $S_n = \sum_{k=1}^n 1_{A_k}$.

(1) Show that $\sum_{n=1}^\infty (1_{A_n} - 1/n)/\log n$ converges a.s.

(2) Show that $S_n/\log n \to 1$ a.s.

Proof. (1) We let $X_n := (1_{A_n} - 1/n)/\log n$. Then $|X_n| \leqslant 1$, $\mathbb{E}X_n = 0$, and

$$\operatorname{var}(X_n) = (\log n)^{-2} \cdot \frac{1}{n} \cdot \frac{n-1}{n} = \frac{n-1}{n^2 \log^2 n} \leqslant \frac{1}{n \log^2 n} = C \cdot \frac{1}{n \log_2^2 n}.$$

By Cauchy condensation test, this series converges since

$$\sum_{n=2}^{\infty} \frac{2^n}{2^n \log_2^2(2^n)} = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, setting $A = 1$ and using Kolmogorov's three-series theorem, we see $\sum_{n=1}^{\infty}(1_{A_n} - 1/n)/\log n$ converges a.s.

(2) Using Kronecker's lemma, since $\log n \to \infty$,

$$\left(S_n - \sum_{k=1}^{n} \frac{1}{n}\right)/\log n \to 0 \text{ a.s.}$$

It is well-known that the limit between the harmonic series and $\log n$ tend to 1, so subtracting gives $S_n/\log n \to 1$ a.s. □

Problem 2

Let $X_1, X_2, \ldots$ be i.i.d. with mean 0 and suppose $\mathbb{E}e^{tX_1} < \infty$ for all $|t| < t_0$. Let γ be the corresponding rate function. Let $b > 0$ and let g be a continuous function which is 0 outside $[0, b]$.

(1) Show that for $a, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(a \leqslant S_n/n < a + \delta) = -\gamma(a).$$

(2) Show that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}(e^{ng(S_n/n)}) \leqslant \sup_{c\in[a,b]} (g(c) - \gamma(c)).$$

Solution. (1) Straight from definition, $P(S_n/n \geqslant a) \geqslant \mathbb{P}(S_n/n \geqslant a + \delta)$, so $-\gamma(a+\delta) \leqslant -\gamma(a)$. Since

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\mathbb{P}(S_n/n \geqslant a+\delta)}{\mathbb{P}(S_n/n \geqslant a)}\right) = \lim_{n\to\infty} \frac{\mathbb{P}(S_n/n \geqslant a+\delta)^{1/n}}{\mathbb{P}(S_n/n \geqslant a)^{1/p}} = e^{-\gamma(a+\delta)+\gamma(a)} < 1$$

(because $-\gamma(a+b) + \gamma(a) < 0$), taking n^{th} power gives

$$\lim_{n\to\infty} \frac{\mathbb{P}(S_n/n \geqslant a+\delta)}{\mathbb{P}(S_n/n \geqslant a)} = 0.$$

Finally, we can put everything together:

$$\begin{aligned}
\lim_{n\to\infty} n^{-1} \log \mathbb{P}(a \leqslant S_n/n < a+\delta) &= \lim_{n\to\infty} n^{-1} \log\left[\mathbb{P}(S_n/n \geqslant a) - \mathbb{P}(S_n/n \geqslant a+\delta)\right] \\
&= \lim_{n\to\infty} n^{-1} \log\left[\mathbb{P}(S_n/n \geqslant a) \cdot \left[1 - \frac{\mathbb{P}(S_n/n \geqslant a+\delta)}{\mathbb{P}(S_n/n \geqslant a)}\right]\right] \\
&= \lim_{n\to\infty} n^{-1} \log \mathbb{P}(S_n/n \geqslant a) + \lim_{n\to\infty} n^{-1} \log\left(1 - \frac{\mathbb{P}(S_n/n \geqslant a+\delta)}{\mathbb{P}(S_n/nh \geqslant a)}\right) \\
&= -\gamma(a) + \log 1 = -\gamma(a).
\end{aligned}$$

Problem 3

Suppose $X > 0$ is integer-valued with $\mathbb{P}(X = k) = q_k e^{-t_0 k}$ where $t_0 > 0$ and $\{q_k\}$ satisfies $k^{-1} \log q_k \to 0$. Let $\varphi(t)$ be the MGF of X.

(1) Find the value $t_{\max} = \sup\{t : \varphi(t) < \infty\}$.

(2) What condition on $\{q_k\}$ is equivalent to $\varphi(t_{\max}) < \infty$?

(3) What condition on $\{q_k\}$ is equivalent to $\lim_{t \uparrow t_{\max}} (\log \varphi)'(t) < \infty$?

Solution. (1) We rewrite the expectation as $\mathbb{E}(e^{tX}) = \sum q_k e^{(t-t_0)k} = \sum e^{k(t-t_0+c_k)}$, where $c_k = k^{-1} \log q_k$. By assumption $c_k \to 0$.

If $t > t_0$, then for sufficiently large N, if $k \geqslant N$, $|c_k| < (t - t_0)/2$, and so

$$\sum_k e^{k(t-t_0+c_k)} \geqslant \sum_{k \geqslant N} e^{k(t-t_0+c_k)} \geqslant \sum_{k \geqslant N} e^{k(t-t_0)/2} = \infty.$$

Similarly, if $t < t_0$, then for sufficiently large N, $|c_k| < (t - t_0)/2$ whenever $k \geqslant N$, and

$$\sum_k e^{k(t-t_0+c_k)} = C + \sum_{k \geqslant N} e^{k(t-t_0)/2} < \infty,$$

since now the tail is a geometric series of ratio $e^{-1/2}$. Therefore $t_{\max} = t_0$.

(2) Since

$$\varphi(t_0) = \sum_k e^{kc_k} = \sum_k e^{\log q_k} = \sum_k q_k,$$

we see $\varphi(t_{\max}) < \infty$ iff $\{q_k\}$ is summable.

(3) Note that $(\log \varphi)'(t) = \varphi'(t)/\varphi(t)$, which is equal to

$$\frac{\sum_{k=1}^{\infty} k e^{k(t-t_0+c_k)}}{\sum_{k=1}^{\infty} e^{k(t-t_0+a_k)}}.$$

Since $k \geqslant 1$, it is clear that $\varphi'(t) \geqslant \varphi(t)$. To obtain a finite limit, both the numerator and the denominator must be finite as $t \uparrow t_0$. We have checked the denominator before, so it suffices to check the numerator. Since $\varphi(t)$ is increasing as $t \uparrow t_0$, it suffices to require $\varphi(t_0) < \infty$. That is,

$$\sum_k k e^{kc_k} = \sum_k k q_k < \infty.$$

And indeed, this is a stronger statement than $\sum q_k < \infty$, so this also ensures that the denominator is finite.

Problem 4

Let $X_1, X_2, \dots$ be independent with all $X_n \geqslant 0$. Show that $\sum_n X_n$ converges iff $\sum_n \mathbb{E}(X_n/(1 + X_n)) < \infty$.

Proof. If $\sum X_n$ converges, by Kolmogorov's $0-1$ law it must converge to a constant random variable, say $X = c$.

Since $X_n/(1+X_n) \leqslant X_n$, we have

$$\mathbb{E}\sum_{n=1}^{\infty} X_n/(1+X_n) \leqslant \mathbb{E}\sum_{n=1}^{\infty} X_n = c.$$

Finally, using MCT we exchange $\mathbb{E}$ with $\sum$ and obtain the claim.

Conversely, we use the Three Series Theorem and consider $X_n 1_{X_n \leqslant 1}$. We have

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > 1) = \sum_{n=1}^{\infty} \mathbb{E}X_n 1_{X_n>1} \leqslant \sum_{n=1}^{\infty} 2\mathbb{E}X_n/(1+X_n)$$

because $x \mapsto x/(1+x)$ is increasing and in particular when $x \geqslant 1$, $x/(1+x) \geqslant 0.5$. We also have

$$\sum_{n=1}^{\infty} \mathbb{E}X_n 1_{X_n \leqslant 1} \leqslant \sum_{n=1}^{\infty} \mathbb{E}\frac{2X_n}{1+X_n} < \infty$$

since in this case $1 + X_n \leqslant 2$. Finally,

$$\sum_{n=1}^{\infty} \operatorname{var}(X_n 1_{X_n \leqslant 1}) = \sum_{n=1}^{\infty} \mathbb{E}X_n^2 1_{X_n \leqslant 1} - (\mathbb{E}X_n 1_{X_n \leqslant 1})^2 \leqslant \sum_{n=1}^{\infty} \mathbb{E}X_n^2 1_{X_n \leqslant 1} \leqslant \sum_{n=1}^{\infty} \mathbb{E}X_n 1_{X_n \leqslant 1} < \infty.$$

Therefore $\sum X_n$ converges almost surely. □

Problem 5

Suppose $X_1, X_2, \ldots$ are i.i.d. with distribution function $F(x) = 1 - e^{-ax^2}$, $a > 0$.

(1) For what values $c > 0$ and $\alpha > 0$ does $\mathbb{P}(X_n > c(\log n)^\alpha \text{ i.o.}) = 1$?

(2) Find constants $\{b_n\}$ such that $\limsup X_n/b_n = 1$ i.o.

Solution. (1) Note that $\sum_n e^{-r\log n} = \sum_n n^{-r}$ which converges iff $r > 1$. To make $\mathbb{P}(X_n > c(\log n)^\alpha \text{ i.o.}) = 1$ it suffices, by second Borel-Cantelli, to ensure the series of probabilities diverge. For convenience, we let $\alpha = 1/2$ so that

$$\mathbb{P}(X_n > c(\log n)^\alpha) = \mathbb{P}(X_n > \exp(-ac^2 \log n)) = n^{-ac^2}.$$

Therefore, if $ac^2 \leqslant 1$, namely, if $c \leqslant a^{-1/2}$, then the sum $\sum_n \mathbb{P}(X_n > c(\log n)^\alpha) = n^{-ac^2} \geqslant \sum 1/n = \infty$, and so $\mathbb{P}(X_n > c(\log n)^\alpha \text{ i.o.}) = 1$.

(2) On the other hand, if $ac^2 > 1$, then the series $\sum_n \mathbb{P}(X_n > c(\log n)^\alpha) = n^{-ac^2} < \infty$, and so $\mathbb{P}(X_n > c(\log n)^\alpha \text{ i.o.}) = 0$. Combining these two results, we define $b_n := a^{-1/2} \cdot (\log n)^{1/2}$, and

$$\begin{cases} \mathbb{P}(X_n > b_n - \epsilon \text{ i.o.}) = 1 \\ \mathbb{P}(X_n > b_n + \epsilon \text{ i.o.}) = 0 \end{cases} \implies \limsup_{n\to\infty} X_n/b_n = 1 \text{ i.o.}$$