MATH 507 Homework 7

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Problem: D3.3.1

Show that if φ is a ch.f. then $\mathfrak{Re}\varphi$ and $|\varphi|^2$ are too.

Proof. Let X have ch.f. φ and let Y be independent from and have the same distribution as $-X$. Then

$$\varphi_{X+Y}(t) = \mathbb{E}(\exp(it(X+Y))) = \mathbb{E}(\exp(itX))\mathbb{E}(\exp(it-X)) = \varphi_X(t)\cdot\overline{\varphi_X(t)} = |\varphi_X(t)|,$$

and from lemma 3.9

$$\varphi_{(X+Y)/2}(t) = \varphi_X(t)/2 + \varphi(Y)/2 = \mathfrak{Re}\varphi_X(t).$$ □

Problem: D3.3.2

Show that

$$\mu(\{a\}) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,\mathrm{d}t.$$

Proof. Let I_T denote the quantity that we take limit on. Then

$$\begin{aligned}
I_T &= \frac{1}{2I}\int_{-T}^{T}\int_{\mathbb{R}} e^{-ita}e^{itx}\,\mathrm{d}\mu(x)\,\mathrm{d}t\\
&= \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}} e^{it(x-a)}\,\mathrm{d}\mu(x)\,\mathrm{d}t\\
[\text{Fubini}] &= \frac{1}{2T}\int_{\mathbb{R}}\int_{-T}^{T} e^{it(x-a)}\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
&= \frac{1}{2T}\int_{\mathbb{R}}\int_{-T}^{T} \cos(t(x-a)) + i\sin(t(x-a))\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
&= \int_{\mathbb{R}}\frac{1}{2T}\int_{-T}^{T} \cos(t(x-a))\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
\left[\text{DCT, as } -1 \leqslant \frac{1}{2T}\int_{-T}^{T}\cos(\cdot) \leqslant 1\right] \to &\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \cos(t(x-a))\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
&= \int_{\{x=a\}}\lim_{T\to\infty}\int_{-T}^{T}\cos(t(x-a))\,\mathrm{d}t\,\mathrm{d}\mu(x) + \int_{\{x\neq a\}}\cdots\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
&= \int_{\{x=a\}} 1\,\mathrm{d}\mu(x) + \int_{\{x\neq a\}}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))\,\mathrm{d}t\,\mathrm{d}\mu(x)\\
&= \mu(\{a\}) + \int_{\{x\neq a\}}\lim_{T\to\infty}\frac{\sin(T(x-a))}{T(x-a)}\,\mathrm{d}\mu(x)\\
&= \mu(\{a\}) + \int_{\{x\neq a\}} 0\,\mathrm{d}\mu(x) = \mu(\{a\}).
\end{aligned}$$ □

Problem: D3.3.8

Show that if X_n and Y_n are independent, X, Y are independent, and $X_n \to X, Y_n \to Y$ weakly, then $X_n + Y_n \to X + Y$ weakly.

Proof. Since $\varphi_{X_n+Y_n} = \varphi_{X_n}\varphi_{Y_n}$ by independence, it suffices to show, according to the continuity theorem, that (i) $\varphi_{X_n}\varphi_{Y_n} \to \varphi_X\varphi_Y$ pointwise, and (ii) $\varphi_X\varphi_Y$ is continuous at 0.

Both are obvious. For (i), $X_n \to X$ weakly implies $\varphi_{X_n} \to \varphi_X$ pointwise; similarly for Y_n and Y. Thus $\varphi_{X_n}\varphi_{Y_n} \to \varphi_X\varphi_Y$ poitnwise. For (ii), note that the continuity theorem implies φ_X is both the ch.f. of X and the limit of φ_{X_n}, and similarly φ_Y is the limit of φ_{Y_n}. As ch.f.'s they are continuous at 0, so $\varphi_X\varphi_Y$ is continuous at 0 as well. This completes the proof. ☐

Problem: D3.3.12

Use 3.3.18 and the series expansion for $e^{-t^2/2}$ to show that the standard normal distribution has

$$\mathbb{E}X^{2n} = (2n-1)!!$$

Proof. We first verify the assumption: $\int_{-\infty}^{\infty} |x| \exp(-x^2/2)\,\mathrm{d}x = 2\int_0^{\infty} x\exp(-x^2/2)\,\mathrm{d}x = 2$, and

$$\int_{-\infty}^{\infty} |x|^n \exp(-x^2/2)\,\mathrm{d}x = 2\int_0^{\infty} x^n \exp(-x^2/2)\,\mathrm{d}x = 2 - x^{n-1}\exp(-x^2/2)\Big|_{x=0}^{\infty} + 2\int_0^{\infty} x^{n-1}\exp(-x^2/2)\,\mathrm{d}x < \infty$$

by induction. By theorem 3.3.18, setting $t = 0$ we have $i^n\mathbb{E}X^n = \varphi^{(n)}(t)\mid_{t=0}$. When differentiating $\varphi(t)$ at $t = 0$, all but the first term (i.e. the one without powers of t) survive, so $\varphi^{(n)}(t)\mid_{t=0} = \exp(-t^2/2)$. Since $e^{-t^2/2} = 1 + \sum_{k=1}^{\infty}(-1)^k t^{2k}/(k!2^k)$, we obtain $\mathbb{E}X^{2n} = (2n)!/(2^n n!) = (2n-1)!!$. ☐

Problem 1

Suppose $X_1, X_2, \ldots$ are i.i.d. with ch.f. $\varphi(t) = \exp(-|t|^\alpha)$ where $0 < \alpha \leqslant 2$. Show that $(X_1 + \ldots + X_n)/n^{1/\alpha}$ has the same distribution as X_1.

Proof. Let Y_i denote $X_i/n^{1/\alpha}$ and $Y = \sum Y_i$. Then

$$\varphi_{Y_i}(t) = \exp(-|t/n^{1/\alpha}|^\alpha) = \exp(-|t|^\alpha/n),$$

so by independence

$$\varphi_Y(t) = \exp(-n|t|^\alpha/n) = \exp(-|t|^\alpha) = \varphi_{X_1}(t).$$ ☐

Problem 2

Suppose $X_1, X_2, \ldots$ are i.i.d. with ch.f. $\varphi(t) = 1 - \beta|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$. Let Z have ch.f. $\psi(t) = \exp(-|t|^\alpha)$ for some $\alpha \in (0, 2]$. Find b, θ such that $S_n/(bn^\theta)$ converges weakly to Z.

Solution. From the previous problem, $S_n/n^{1/\alpha}$ has ch.f. $1-\beta|t|^\alpha + o(|t|^\alpha)$. Setting b as β^α and $\theta = 1/\alpha$ gives

$$\mathbb{E}\exp(it(S_n/n^\alpha \cdot b^{-1})) = \mathbb{E}\exp(itX_1/b) = \varphi_{X_1}(t/b) = 1-\beta|\beta^{1/\alpha}t|\alpha + o(|t|^\alpha) = 1-|t|^\alpha + o(|t|^\alpha)$$

which is what we seek.

Problem 3

Suppose $\alpha \in (0,1)$, and X satisfies $\mathbb{E}|X|^\alpha < \infty$ with ch.f. φ. Show that $|1-\varphi(t)| = o(|t|^\alpha)$ as $t \to 0$.

Problem 4

(1) Let $X_1, X_2, \ldots$ be i.i.d. with ch.f. φ and suppose $\varphi'(0) = ia$ for some a. Show $S_n/n \to a$ in probability.

(2) The converse is also true: if $S_n/n \to a$ in probability then $\varphi'(0) = ia$. Give an example of a r.v. X for which $\mathbb{E}|X| = \infty$ but $\varphi'(0)$ exists.

Proof. (1) We write the ch.f. of S_n/n as $\varphi_{S_n/n}(t) = \varphi(t/n)^n$. Since convergence in probability to a constant is equivalent to weak convergence, it suffices to show $S_n/n \to a$ in distribution. Since

$$\lim_{n\to\infty} \varphi(t/n)^n = \lim_{n\to\infty}\left(1+\frac{n(\varphi(t/n)-1)}{n}\right)^n,$$

and

$$\varphi'(0) = \lim_{n\to\infty}\frac{\varphi(t/n)-\varphi(0)}{t/n} = \lim_{n\to\infty}\frac{\varphi(t/n)-1}{t/n} = ia \implies \lim_{n\to\infty} n(\varphi(t/n)-1) = iat,$$

the hint implies that $\varphi(t/n)^n \to \exp(ita)$, which completes the proof.

(2) One example is Durrett exercise 2.2.4 or HW3 problem 1. □

Problem 5

X is a lattice random variable if there is a set of form $\{a+bk : k \in \mathbb{Z}\}$ with $a \in \mathbb{R}, b > 0$ such that all values of X lie in the set. Suppose X has ch.f. $\varphi(t)$ and $|\varphi(t)| = 1$ for some $t \neq 0$. Show X is a lattice r.v.

Proof. Let θ be such that $\varphi(t) = e^{i\theta}$. Then $\varphi_{X-\theta/t}(t) = \mathbb{E}(\exp(it(X-\theta/t))) = \mathbb{E}\exp(itX)/e^{i\theta} = 1$. This means that $\exp(it(X-\theta/t)) = 1$ almost surely; namely, $X = 2k\pi + \theta/t$ almost surely. That is, the values of X lie in $\{\pi/t + 2\pi\cdot k\}$ almost surely. Hence X is a lattice r.v. □

Problem 6

Suppose X_n, $1 \leqslant n \leqslant \infty$ are random variables with ch.f.'s φ_n, all dominated by $g \in L^1$. If $\varphi_n \to \varphi_\infty$ pointwise, show that X_n and X_∞ have densities f_n and f_∞ such that $f_n \to f_\infty$ uniformly.

Proof. Theorem 3.3.14 implies that, since $\varphi_n \in L^1$, X_n has density $f_n(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-itx}\varphi(x)\,\mathrm{d}y$. Then since

φ_n's are dominated by g, $|\varphi_n - \varphi| \leqslant 2g$ which is still integrable, and $|\varphi_n(t) - \varphi(t)| \to 0$ pointwise. By DCT,

$$\begin{aligned}
\sup_{x\in\mathbb{R}}|f_n(x) - f(x)| &= \sup\left|\int_{\mathbb{R}} e^{-itx}\varphi_n(x)\,\mathrm{d}x - \int_{\mathbb{R}} e^{-itx}\varphi(x)\,\mathrm{d}x\right| \\
&\leqslant \sup\int_{\mathbb{R}}|e^{-itx}(\varphi_n(x) - \varphi(x))|\,\mathrm{d}x \\
&= \sup\int_{\mathbb{R}}|\varphi_n(x) - \varphi(x)|\,\mathrm{d}x \\
&\to \int 0\,\mathrm{d}x = 0,
\end{aligned}$$

which concludes the proof. □

Problem 7

If X, Y are independent Gaussians with mean zero and variance σ_1^2 and σ_2^2, show $X + Y \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

Proof. Since a standard Gaussian has ch.f. $\exp(-t^2/2)$ and $X = \sigma_1$ times a standard Gaussian,

$$\varphi_X(t) = \varphi(\sigma_1 t) = \exp(-t^2\sigma_1^2/2),$$

and similarly $\varphi_Y(t) = \exp(-t^2\sigma_2^2/2)$. By independence, $\varphi_{X+Y}(t) = \exp(-t^2(\sigma_1^2 + \sigma^2)/2)$, corresponding to $\mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$. □