MATH 507a Homework 8

Qilin Ye

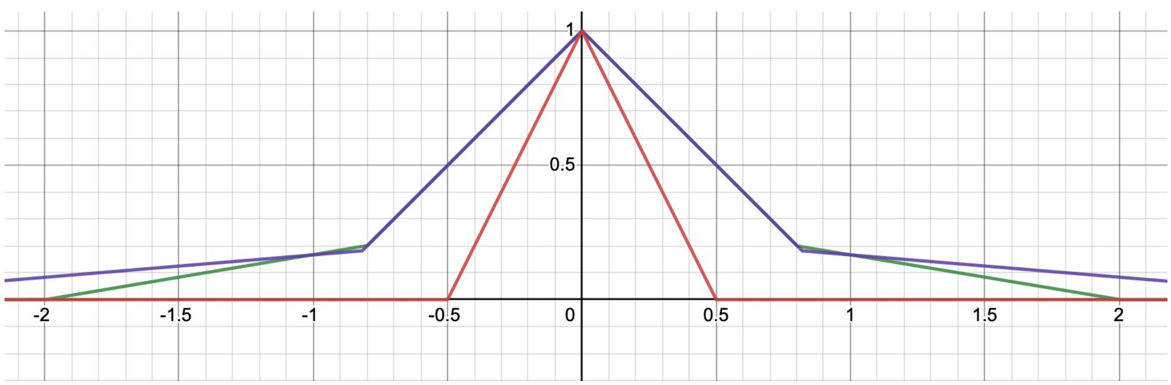
November 21, 2022

Durrett 3.3.22

Solution. Let $f(t) = \max\{1 - |t|, 0\}$. We have shown in lecture (and by Polya's criterion) this is a ch.f. (of $(1 - \cos x/(\pi x^2))$). We define

$$\varphi_X(t) = f(2t), \qquad \varphi_Y(t) = \max\{f(t), f(t/2)/3\}, \qquad \text{and } \varphi_Z(t) = \max\{f(t), f(t/3)/4\}.$$

These numbers are somewhat picked arbitrarily; for a plot, see below (red φ_X, green φ_Y, and purple φ_Z):



All of them vanish outside a bounded interval and all of them is continuous and equals 1 at $t = 0$. By Polya's criterion they are ch.f.'s. Let X, Y, Z be variables corresponding to these three ch.f.'s and let them be independent. Since $\varphi_Y = \varphi_Z$ on $[-0.5, 0.5]$ and φ_X vanishes outside this interval, $\varphi_Y\varphi_Z = \varphi_X\varphi_Z$ which, by independence, suggests $X+Y$ and $X+Z$ have the same distribution. Clearly, $\varphi_Y \neq \varphi_Z$, so Y, Z do not have the same distribution. This completes the proof. □

Durrett 3.3.23

Proof. Since $X + Y \stackrel{d}{=} X$, $\varphi_X\varphi_Y \equiv \varphi_X$. Since φ_X is continuous at origin and equals 1, there exists $\epsilon > 0$ such that $\varphi_X > 0$ on $(-\epsilon, \epsilon)$. For $t \in (-\epsilon, \epsilon)$ this implies $\mathbb{E}\exp(itY) = 1$ so in particular $\cos(tY) = 1$ a.s. Therefore $tY \in 2\pi\mathbb{Z}$ a.s., and $Y \in 2\pi\mathbb{Z}/t$ a.s., for all $t \in U$. By picking $t_1 \in U$ nonzero rational and t_2 irrational, we see $Y \in 2\pi\mathbb{Z}/t_1 \cap 2\pi\mathbb{Z}/t_2$ a.s., but this intersection is precisely $\{0\}$. □

Durrett 3.3.24

Proof. Since $X \leqslant \lambda$ with probability 1, $\nu_k \leqslant \mathbb{E}|X|^k \leqslant \lambda^k$. Conversely, for $\epsilon > 0$ we have $\mathbb{P}(|X| < \lambda - \epsilon) < 1$ so $\mathbb{P}(|X| \geqslant \lambda - \epsilon) > 0$, and

$$\nu_k = \mathbb{E}|X|^k \geqslant (\lambda - \epsilon)^k \mathbb{P}(|X| \geqslant \lambda - \epsilon)$$

by Markov, which implies $\liminf \nu_k^{1/k} \geqslant (\lambda - \epsilon)\mathbb{P}(|X| \geqslant \lambda - \epsilon)^{1/k} = \lambda - \epsilon$. □

Durrett 3.4.1

Solution. Each roll X_i can be viewed as a Bernoulli with parameter $1/6$. The mean is $1/6$ and variance $5/36$. If $n = 180$ and $S_n = \sum_{i=1}^n 1_{X_i=6}$ then

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \to \mathcal{N}(0,1);$$

in this case, $S_n \to \mathcal{N}(30, \sqrt{180} \cdot \sqrt{5/36}) = \mathcal{N}(30, 5)$. Therefore with continuity correction

$$\mathbb{P}(S_n \leqslant 24.5) = \mathbb{P}(\mathcal{N}(0,1) \leqslant -1.1) = \Phi(-1.1) \approx 0.1357.$$

Durrett 3.4.5

Proof. Note that $\operatorname{var}(X_1) = \mathbb{E}X_1^2 = \sigma^2$, so

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \to \sigma^2$$

in probability by law of large numbers. In particular $n^{-1}\sum X_i^2 \to \sigma^2$ in distribution, and so $(\sum X_i^2)^{1/2} \to \sigma\sqrt{n}$ in distribution. Using CLT and (A) below, we see that $S_n/(\sum X_i)^{1/2}$ converges in distribution to a standard normal. □

(A)

Proof. By theorem 3.2.8 there exist Y_n, Y with distribution functions X_n, X such that $Y_n \to Y$ a.s. Clearly, $c_n \to c$ a.s., so $c_n Y_n \to cY$ a.s. and so $c_n Y_n \to cY$ in distribution. Since $c_n Y_n$ and $c_n X_n$ have the same distribution and likewise for cY, cX, we have thus shown $c_n X_n \to cX$ in distribution. □

(B)

Solution. We replace the notation of A_k by $A_{n,k}$ to stress n, the total number of elements to permute on. Observe that N_n is simply the sum of indicators of A_i, i.e., $N_n = \sum_{k=1}^n \mathbf{1}[A_{n,k}]$. This implies

$$\mathbb{E}N_n = \sum_{k=1}^n \mathbb{E}\mathbf{1}[A_{n,k}] - \sum_{k=1}^n \frac{1}{n-k+1} \sim \log n,$$

and

$$\operatorname{var}\mathbb{E}N_n = \sum_{k=1}^n ((n-k+1)^{-1} - (n-k+1)^{-2}) \sim \log n$$

as well. Therefore, if we let $a_n = \log n, b_n = \sqrt{n}$, and define

$$X_{n,m} = \frac{\mathbf{1}[A_{n,k}] - (n-k+1)^{-1}}{\sqrt{\log n}}$$

then

- $\mathbb{E}X_{n,m} = 0$,
- $\sum_{m=1}^n \mathbb{E}X_{n,m}^2 \to 1$, and

- $\sum_{m=1}^n \mathbb{E}(X_{n,m}^2 \mathbf{1}[|X_{n,m}| > \epsilon]) \to 0$ as $\mathbb{P}(|X_{n,m}| > \epsilon) = 0$ for n sufficiently large.

Using Lindeberg-Feller we conclude that $(N_n - \log n)/\sqrt{\log n}$ converges in distribution to a standard Gaussian.

(C)

Proof. If X, X' are i.i.d. then

$$\varphi_{X-X'}(t) = \varphi_X(t)\varphi'_X(-t) = \varphi_X(t)\varphi_X(-t) = |\varphi_X(t)|^2 \geqslant 0.$$

If $X - Y$ were to be uniform on some interval it must be symmetric across 0, for otherwise the i.i.d. violation is violated. If $X - Y$ is uniform on $(-a, a)$ for some a, then the ch.f. is

$$\frac{1}{a}\int_{-a}^{a} e^{itx}\,\mathrm{d}x = \frac{\sin(at)}{at},$$

which not non-negative. Contradiction. □

(D)

Proof. We begin by noting $\mathbb{E}X_k = 0$ and $\mathbb{E}X_k^2 = k^{2\alpha-\beta}$. Using the identity $\sum_{j=1}^n j^p = n^{p+1}/(p+1)$ which follows from $\int_0^1 x^m\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n}(k/n)^m/n$, we obtain

$$\mathrm{var}(S_n) = n^{2\alpha-\beta+1}/(2\alpha-\beta+1).$$

Setting $b_n := n^{\alpha-(\beta-1)/2}$, we then have $\mathrm{var}(S_n/b_n) = 1/(2\alpha-\beta+1)$. It remains to verify

$$\lim_{n\to\infty}\sum_{m=1}^{n}\mathbb{E}(|X_m/n^{\alpha-(\beta-1)/2}|^2 : |\ldots| > \epsilon) = \lim_{n\to\infty}\sum_{m=1}^{n}\mathbb{E}(|X_m|^2/n^{2\alpha-\beta+1}\mathbf{1}_{|X_m|>\epsilon n^{2\alpha-\beta+1}}) \to 0.$$

If $\alpha < 0$, then $\alpha-\beta+1 > (\beta-1)/2-(\beta-1) = (1-\beta)/2 > 0$, so $n^\alpha > n^{2\alpha-\beta+1}$. If $\alpha \geqslant 0$, then $2\alpha-\beta+1 \geqslant 2\alpha$ and again $n^\alpha > n^{2\alpha-\beta+1}$. Therefore all but finitely terms of form $|X_m|^2/n^{2\alpha-\beta+1}\mathbf{1}_{|X_m|>\epsilon n^{2\alpha-\beta+1}}$ remain, and as $n \to \infty$, the sum indeed converges to 0. This establishes Lindeberg-Feller and we are done, with $b_n = n^{\alpha-(\beta-1)/2}$ and limiting distribution $\mathcal{N}(0,1)/(2\alpha-\beta+1)$. □

(E)

$$\begin{aligned}
\mathbb{P}(X = m) &= \sum_{n=0}^{\infty}\mathbb{P}(X = m, N = n) = \sum_{n=m}^{\infty}\mathbb{P}(\sum_{k=1}^{n}\xi_k = m, N = n)\\
&= \sum_{n=m}^{\infty}\mathbb{P}(N = n)\mathbb{P}(\sum_{k=1}^{n}\xi_k = m)\\
&= \sum_{n=m}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\cdot\binom{n}{m}p^m(1-p)^{n-m}\\
&= \sum_{n=m}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\frac{n!}{m!(n-m)!}p^m(1-p)^{n-m}\\
&= \frac{e^{-\lambda}p^m\lambda^m}{m!}\sum_{n=m}^{\infty}\frac{\lambda^{n-m}(1-p)^{n-m}}{(n-m)!}\\
&= \frac{e^{-\lambda}p^m\lambda^m}{m!}\cdot e^{\lambda(1-p)} = \frac{e^{-\lambda p}(\lambda p)^m}{m!}.
\end{aligned}$$