Multivariate Normal

Beginning of Nov. 18, 2022

Let $X = (X_1, ..., X_\ell)$ be a random vector with $\text{var}(X_i) < \infty$ and covariance matrix $\Sigma_{i,j} = \text{cov}(X_i, X_j)$. Then for any vector θ,

$$\text{var}(\theta \cdot X) = \text{var}(\sum_{i=1}^{\ell} \theta_i X_i) = \sum_{i,j} \theta_i \theta_j \,\text{cov}(X_i, X_j) = \theta^T \Sigma \theta \in \mathbb{R}.$$

This shows Σ is PSD and symmetric. If T is a linear transformation of X, then

$$\text{cov}((TX)_i, (TX)_j) = \text{cov}(\sum_k T_{i,k} X_k + \sum_\ell T_{j,\ell} X_\ell) = (T\Sigma T^T)_{i,j}.$$

From linear algebra, since Σ is PSD, there exists an unitary U ($U^{-1} = U^*$ and orthonormal) such that $U^T \Sigma U$ is diagonal. From our remark above, viewing U^T as a linear transformation, the resulting random vector $U^T X$ has uncorrelated components.

If X has density f_X and T a multivaraible linear transformation, then TX has density

$$f_{TX}(x) = \frac{1}{|\text{det} T|} f_X(T^{-1}x).$$

Finally, we are ready to talk about multivariate normal distribution.
Of course, the standard multivariate normal has each coordinate as an independent $\mathcal{N}(0,1)$. The density

$$f(X) = (2\pi)^{-d/2} \exp(-\sum_{i=1}^{d} x_i^2/2) = (2\pi)^{-1/2} \exp(-x^T I x/2) =: \mathcal{N}(0, I).$$

If T is invertible then

$$f_{TX}(x) = (2\pi)^{-d/2} |\text{det} T|^{-1} \exp(-x^T T^{-T} T^{-1} x/2) = (2\pi)^{-d/2} |\text{det} T|^{-1} \exp(-x(TT^*)^{-1} x/2).$$

This gives rise to a more general multivariable normal $\mathcal{N}(\mu, \Sigma)$, whose density is

$$f(x) = (2\pi)^{-d/2} ||\Sigma|^{-1/2} \exp(-(x-\mu)^T \Sigma^{-1} (x-\mu)/2)$$

where $\mu \in \mathbb{R}^k$ and Σ PSD.