Beginning of Nov. 21, 2022

For a degenerate mnultivariate normal, consider $\mu = 0$ and $r < d$ (rank). We take $X_1, ..., X_r \sim \mathcal{N}(0, \tilde{S})$ where $\tilde{S}$ is invertible. we put $X = (X_1, ..., X_r, 0, ..., 0)$ corresponding to block diagonal $\tilde{S}, 0$. Then TX has covariance matrix TST^T. Given Σ, we want to choose $\tilde{S}, T$ so that $TST^T = \Sigma$.
We know there exists a unitary matrix T with $T\Sigma T$ = diagonal $(\lambda_1^2, ..., \lambda_r^2, 0, ...)$. Let $\tilde{D}$ = diagonal $(\lambda_1^2, ..., \lambda_r^2)$ and let $\tilde{X} \sim \mathcal{N}(0, \tilde{D})$, $X = (\hat{X}, 0, ..., 0)$. Then TX has covariance matrix TDT^T since $T^T = T^{-1}$.

Proposition

If $X \sim \mathcal{N}(\mu, \Sigma)$ with Σ nonsingular, then the marginals X_i are normal.

Proof. WLOG $\mu = 0$ and we are looking at the first coordinate, X_1.
The claim is easy if Σ has first row $(\sigma_1^2, 0, ..., 0)^T$ and column $(\sigma_1^2, 0, ..., 0)$. In this case,

$$f_X(x) = C\exp(-x^T\Sigma^{-1}x/2) = C\exp(-\frac{x_1^2}{2\sigma_1^2} - g(x_2, ..., x_d)).$$

Therefore

$$\begin{aligned} f_{X_1}(x) &= \int_{x_2,...,x_d} f_X(x_1, ..., x_d)\, \mathrm{d}x_2 ... x_d \\ &= \text{Const}\exp(-\frac{x_1^2}{2\sigma_1^2}). \end{aligned}$$

Since f_{X_1} integrates to 1 the constant must match up, so $X_1\mathcal{N}(0, \sigma_1^2)$.
For the general case, we need to find T so that TX's covariance has the special form with $(TX)_1 = X_1$.
We take unitary U such that the 1^{st} row of U is perpendicular to the j^{th} column of $\Sigma^{-1/2}$ $j \geqslant 2$. Take $T = U\Sigma^{-1/2}$.
Then TX has covariance matrix

$$T\Sigma T^T = U\Sigma^{-1/2}\Sigma\Sigma^{-1/2}U^T = UU^T = I.$$

[To be fixed] □

Example. Multivariate normal implies normal marginals, but not the converse. For example let $X = \mathcal{N}(0, 1)$ and $\xi = \pm 1$ with probably 0.5 each, independent of X.
Let $Y = (X, \xi X)$, so it's on either diagonal with probability 0.5. Clearly Y is not a bivariate normal, even if its covariance matrix is I.

If $X, X_2, ..., X_d$ are independent $\mathcal{N}(\mu_i, \sigma_i^2)$, then $X = (X_1, ..., X_d) \sim \mathcal{N}(\mu, \Sigma)$ with Σ = diagonal $(\sigma_1^2, ..., \sigma_d^2)$. Since

$$f_X(x) = \prod_{i=1}^d f_{X_i}(x_i) = \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_i}\exp(-x_i/(2\sigma_i^2)) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}\exp(-x^T\Sigma^{-1}x/2),$$

we obtain the following result:

Proposition

While not necessarily true for other distributions, for $(X_1, ..., X_n)$ multivariate normal, X_i's are uncorrelated iff X_i's are independent.

(The previous example we have shows that if (X_1, X_2) is not multivariate normal, even if it has normal marginals, then (uncorrelated but dependent) can happen.)

If $X \sim \mathcal{N}(0, \Sigma)$, and T is invertible, then TX has density

$$f_{TX}(x) = \frac{1}{|T|} f_X(T^{-1}x) = \frac{1}{(2\pi)^{d/2}|T||\Sigma|^{1/2}} \exp(x^T T^{-T} \Sigma^{-1} T^{-1} x/2)$$
$$= \frac{1}{(2\pi)^{d/2}|T\Sigma T^T|^{1/2}} \exp(-x^T (T\Sigma T^T)^{-1} x/2) \sim \mathcal{N}(0, T\Sigma T^T).$$

Therefore the marginals are normal, in particular $(TX)_1$. Since T is arbitrary, $\theta \cdot X$ is normal for all $\theta \in \mathbb{R}^d$, with $\operatorname{var}(\theta \cdot X) = \theta^T \Sigma \theta$.
More generally, if $X \sim \mathcal{N}(\mu, \Sigma)$ and T is invertible then $TX \sim \mathcal{N}(T\mu, T\Sigma T^T)$.

Characteristic functions of $\mathcal{N}(\mu, \Sigma)$

$$\varphi_X(\theta) = \mathbb{E}e^{i\theta \cdot X} = \varphi_{\theta \cdot X}(1) = \exp(-\operatorname{var}(\theta \cdot X)/2) = \exp(-\theta^T \Sigma \theta/2).$$