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Axiomatic setup of probability

(1) A **universal set** Ω representing outcomes of an "experiment,"

(2) A σ-field (σ-algebra) $\mathcal{F}$ of subsets of Ω (typically not the power set of Ω, especially when Ω is uncountable), called **events**, for which probabilities are defined†, and

† A **field** $\mathcal{F}$ is a nonempty collection of sets that is closed under *finite* union:

- If $A, B \in \mathcal{F}$ then $A \cup B, A^c \in \mathcal{F}$ (so $A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$).

† A **σ-field** is in addition closed under *countable* union. Consequences:

- $A_1, A_2, \ldots \in \mathcal{F}$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$, $\Omega = A \cup A^c \in \mathcal{F}$, and $\varnothing = \Omega^c \in \mathcal{F}$.

(3) A **probability measure** (p.m.) $\mathbb{P}$, a measure on $\mathcal{F}$, i.e., $\mu : \mathcal{F} \to [0, \infty]$, with

- $\mu(A) \geqslant \mu(\varnothing) = 0$,
- If $A_1, A_2, \ldots \in \mathcal{F}$ are pairwise disjoint, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, and
- $\mu(\Omega) = 1$.

(4) We call the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ a **probability space**. Examples:

- Let $\Omega := \mathbb{Z}$, $\mathcal{F}$ the power set of Ω, and

$$p(n) := \begin{cases} 2^{-n} & n \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathbb{P} : \Omega \to [0, 1]$ by $\mathbb{P}(A) := \sum_{a \in A} p(a)$.

Generation of σ-fields

Many σ-fields of interest are generated by some collection $\mathcal{G}$ of events, so it suffices to simply define probability on $\mathcal{G}$ first. We define $\sigma(\mathcal{G})$, the σ-field **generated** by $\mathcal{G}$, to be the intersection of all σ-fields containing $\mathcal{G}$ (this is indeed a σ-field). Right from this definition we have $\mathcal{G} \subset \sigma(\mathcal{G}) \subset \mathcal{F}$.

Examples: if $\mathcal{G}$ is the collection of all open sets in Ω, then $\sigma(\mathcal{G})$ denotes the collection of **Borel sets** in Ω. In $\mathbb{R}$, the Borel sets can be generated by all intervals, whether open or closed or neither or both. Borel sets in $\mathbb{R}$ can be wild, for example $\mathbb{Q}$ or

$$E = \{x \in [0, 1] : x \text{ does not have } 8 \text{ in its decimal expansion}\}.$$

Question: we define measures of intervals to be their lengths, but how about such sets?