# Chapter 1

# Random Variables

## 1.1 Random Variables

Beginning of Aug. 29, 2022

Recall $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ is **measurable** if $\{X \in A\} := \{\omega \in \Omega : X(\omega) \in \mathcal{A}\} \in \mathcal{F}$ for all $A \in \mathcal{S}$.

Under this context, we define a **random variable** (r.v.) to be a $\mathbb{R}$-valued measurable function on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Lemma**

The collection $\{A \in \mathcal{S} : \{X \in A\}$ is measurable$\}$ is a $\sigma$-field (so it is contained in $\mathcal{S}$).

In particular, $X$ is measurable if and only if the $\sigma$-field mentioned above is *all* of $\mathcal{S}$. By the general principle, if this $\sigma$-field contains a set of generators of $\mathcal{S}$ then it must contain $\mathcal{S}$ and therefore equals $\mathcal{S}$.

For example, $(\mathbb{R}, \mathcal{R})$ is generated by the collection of $h$-intervals $\{(-\infty, x] : x \in \mathbb{R}\}$. Therefore it suffices to require $\{X \in x\} \in \mathcal{F}$ for all $x$ for $X$ to be a random variable.

More generally, if $\Omega$ is a topological space, $\mathcal{F}$ the Borel sets, and $X : \Omega \to \mathbb{R}$ continuous, then since $(-\infty, x]$ is closed, $\{X \leqslant x\}$ is closed hence Borel, and $X$ is a r.v.

In this measure-theoretic context, we allow r.v.'s to take values $\pm\infty$, in which case we call them **extended r.v.** We also define a **random vector** to be a measurable function $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^n, \mathcal{R}^n)$.

**Theorem: (D1.3.4)**

Suppose $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $f : (S, \mathcal{S}) \to (T, \mathcal{T})$ are measurable maps, then their composition $f(X) : (\Omega, \mathcal{F}) \to (T, \mathcal{T})$ is measurable. In particular, compositions of random variables are still random variables.

*Proof.* If $B \in \mathcal{T}$ then $f^{-1}(B) \in \mathcal{S}$, so $\{f(X) \in B\} = \{X \in f^{-1}(B)\} \in \mathcal{F}$. □

**Theorem: (D1.3.5)**

If $X_1, \ldots, X_n$ are random variables and $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, then $f(X_1, \ldots, X_n)$ is a random variable. (Note this claim is nontrivial even when each $X_i$ is a r.v.)

*Proof.* To use 1.3.4 we need to verify that $(X_1, ..., X_n)$ is measurable into $\mathbb{R}^n$. By the theorem on generators (1.3.1), it suffices to consider generators of the product $\sigma$-field which in turn suffices to consider products of generators of each $\sigma$-field. That is, we only need to consider

$$\{(X_1, ..., X_n) \in (a_i, b_i) \times ... \times (a_n, b_n)\}$$

which is simply

$$\prod_{i=1}^{n} \{X_i \in (a_i, b_i)\} \in \mathcal{F}.$$ □

## 1.2 Distributions

A random variable $X$ on any $(\Omega, \mathcal{F}, \mathbb{P})$ has a **distribution** $\mu = \mu_X$ inducted on $(\mathbb{R}, \mathcal{R})$ (the push-forward measure of $\mathbb{P}$) given by

$$\mu(A) := \mathbb{P}(X \in A) \qquad \text{for all } A \in \mathcal{R}.$$

The **distribution function** (d.f.) of $X$ (or $\mu$) is then

$$F(x) := \mu((-\infty, x]) = \mathbb{P}(X \leqslant x).$$

We say a d.f. $F$ has **density** $f$ if, for some function $f$,

$$F(x) := \int_{-\infty}^{x} f(t)\, \mathrm{d}t \qquad \text{for all } x.$$

The immediate result following above is that $f \geqslant 0$ and that $\int_{-\infty}^{\infty} f(t)\, \mathrm{d}t = 1$.

*525a recap.* If $F$ is differentiable then $F$ has density $f = F'$. However, if $F$ is differentiable a.e. then the above is *not* necessarily true — consider the cantor function. (We need $F$ to be absolutely continuous.)

We can then also define another measure $Q$ on $(\mathbb{R}, \mathcal{R})$ by

$$Q(A) := \int_A f(t)\, \mathrm{d}t \qquad \text{for all } A \in \mathcal{R}.$$

Note that $\mu, Q$ ahve the same d.f. so $\mu = Q$, which tells us

$$\mathbb{P}(X \in A) = \int_A f(t)\, \mathrm{d}t \qquad \text{for all } A \in \mathcal{R},$$

not just intervals of form $(-\infty, x]$.

**Example: Examples of distributions.**

- Standard normal: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in \mathbb{R}$, also written as $\mathcal{N}(0,1)$.
- If $X \sim \mathcal{N}(0,1)$ then $\mu + \sigma X \sim \mathcal{N}(\mu, \sigma^2)$ with density $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x-\mu)^2/(2\sigma^2))$.
- Chi-squared (with 1 degree of freedom): if $X \sim \mathcal{N}(0,1)$ then $X^2$ has d.f.

$$\mathbb{P}(X^2 \leqslant x) = \mathbb{P}(-\sqrt{x} \leqslant X \leqslant \sqrt{x}) = F(\sqrt{x}) - F(-\sqrt{x})$$

where $F$ is the d.f. for $\mathcal{N}(0,1)$. Then the density of $X^2$ is

$$\begin{aligned} f_{X^2}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(F(\sqrt{x}) - F(-\sqrt{x})) \\ &= \frac{1}{2\sqrt{x}} f(\sqrt{x}) + \frac{1}{2\sqrt{x}} f(-\sqrt{x}) = \frac{f(\sqrt{x})}{\sqrt{x}} = \frac{1}{\sqrt{2\pi x}} e^{-x/2}. \end{aligned}$$

- Exponential: $f(x) = \lambda e^{-\lambda x}$ for $x \geqslant 0$ and some parameter $\lambda > 0$.
- Uniform on $[a,b]$: $f(x) = (b-a)^{-1} 1_{x \in [a,b]}$.

- If $X$ has density $f$ them $\mathbb{P}(X = x) = \int_{\{x\}} f(t)\,\mathrm{d}t = 0$ for all $x$.
- The **left limit**

$$\begin{aligned} F(x-) &= \lim_{n\to\infty} \mathbb{P}(X \leqslant x - 1/n) \\ &= \mathbb{P}(\bigcup_{n\geqslant 1} \{x \leqslant x - 1/n\}) = \mathbb{P}(X < x), \end{aligned}$$

  so the **jump** at $x$ is

$$F(x+) - F(x-) = F(x) - F(x-) = \mathbb{P}(X = x).$$

  That is, if we have a density then there is no jump.
- There can only be countably many points with $\mathbb{P}(X = x) > 0$, i.e., jumps.

**Example: Uniform distribution on Cantor Set $\mathcal{C}$.** Let $\mathcal{C}$ be the Cantor set on $[0,1]$ and let $F$ be the Cantor function on $[0,1]$.