Chapter 1

Laws of Large Numbers

1.1 Independence

Beginning of Sept.12, 2022

First, some definitions/recaps on independence of events and σ-fields:

- Independence of two events: we say events A, B are independent, $A \perp B$, if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- Independence of two σ-fields: $\mathcal{F}$, $\mathcal{G}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{F}$ and $\mathcal{B} \in \mathcal{G}$.
- For *more than* 2 *events*: $A_1, \dots, A_n$ are **mutually independent** if

$$\mathbb{P}(\bigcap_{i \in I} A_i) = \prod_{i \in A} \mathbb{P}(A_i) \qquad \text{for all } I \in \{1, \dots, n\}. \tag{*}$$

- Similarly, for more than 2 sigma fields, we say $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent if the above product identity holds for all $A_i \in \mathcal{A}_i$.
- We say events $A_1, \dots, A_n$ are **pairwise independent** if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \qquad \text{for all } i \neq j.$$

Example: $\mathbb{P}(\bigcap A_i) = \prod \mathbb{P}(A_i)$ is insufficient. Consider two coin tosses. Let $A := \{\text{first is head}\}$, $B := \{\text{second is head}\}$, and $C := \{\text{both tosses are the same}\}$. Then $A \cap B \subset C$, so they are not mutually independent, but

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{8}.$$

In fact, we also have A, B, C pairwise independent here.

- For an *infinite sequence* of A_i's, we say they are independent if (*) holds for any *finite* $I \subset \mathbb{N}$.

Moving to independence of two random variables:

- Two random variables X, Y are independent if

$$\mathbb{P}(X \in Y, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) \tag{**}$$

for all A, B in their corresponding σ-fields. It can be shown that this definition is equivalent to requiring $\sigma(X)$ and $\sigma(Y)$ to be independent.

- To show independence, it is sufficient to check (**) for $(-\infty, x] \times (-\infty, x]$ for all x, y. That is,

$$F_{(X,Y)}(x, y) = F_X(x)F_Y(y) \qquad \text{for all } x, y.$$

Example: $\mathcal{A} \perp \mathcal{B}$ does not imply $\sigma(\mathcal{A}) \perp \sigma(\mathcal{B})$. (The example given in lecture relies heavily on drawings so I will replace it with one easier to type in LaTeX.) Let $\mathcal{A} := \{\{1,2\},\{3,4\}\}$ and let $\mathcal{B} := \{\{2,4\}\}$. Then $\{2,4\} \in \sigma(\mathcal{A})$.

Definition: π-system and λ-system

A collection $\mathcal{G}$ is called a **π-system** if it is nonempty and closed under finite intersections (two suffice):

- $\mathcal{G} \neq \varnothing$, and
- For $A, B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$.

A collection $\mathcal{G}$ is called a **λ-system** if $\mathcal{G}$ contains Ω, is closed under set subtraction, and is closed under countable increasing union:

- $\Omega \in \mathcal{G}$,
- If $A \subset B$ and $A, B \in \mathcal{G}$ then $B \backslash A \in \mathcal{G}$, and
- If $A_n \in \mathcal{G}$ and $A_n \uparrow A$ then $A \in \mathcal{G}$.

The **$\pi - \lambda$ theorem** states that if $\mathcal{P}$ is a π-system and $\mathcal{L}$ a λ-system with $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
We will skip the proof and directly use the result to prove the following (the proof of which we again omit):

Theorem: D2.1.7

If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent σ-fields and each $\mathcal{A}_i$ a π-system, then the $\sigma(\mathcal{A}_i)$'s are independent.

We now discuss the independence of functions of random variable in greater generality. Suppose we have an array of independent random varaibles

$$\{X_{i,j} : i \leqslant n, j \leqslant m(i)\}$$

and n functions

$$X_{1,1}, \ldots, X_{1,m(1)} \mapsto f_1(X_{1,1}, \ldots, X_{1,m(1)})$$
$$X_{2,1}, \ldots, X_{2,m(2)} \mapsto f_2(x_{2,1}, \ldots, X_{2,m(2)})$$

and so on, where each $f_i : \mathbb{R}^{m(i)} \to \mathbb{R}$. **Question**: are these random variables $f_i(\cdot)$ independent? The answer is yes, and we will formulate the question in terms of σ-fields:

Theorem: D2.1.10

Given an independent collection of σ-fields $\{\mathcal{F}_{i,j} : i \leqslant n, j \leqslant m(i)\}$, let $\mathcal{B}_i := \sigma(\mathcal{F}_{I,1}, \ldots, \mathcal{F}_{i,m(i)})$ (i.e., the i^{th} row listed above). Then $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are independent.

Proof. For each row, let

$$A_i := \{\text{all} \bigcap_{i=1}^{m(i)} A_{i,j} \text{ with } A_{i,j} \in \mathcal{F}_{i,j}\}.$$

Then $\mathcal{A}_i$ is a π-system that contains Ω (intersection of all $\Omega \in \mathcal{F}_{i,j}$) and also all $\mathcal{F}_{i,j}$ (intersection of $\mathcal{F}_{i,j}$ with a bunch of Ω's). Therefore $\mathcal{A}_i$ generates $\mathcal{B}_i$. Finally, the $\mathcal{A}_i$'s are independent:

$$\mathbb{P}(\bigcap_{i=1}^{n}(\bigcap_{j=1}^{m(i)} A_{i,j})) = \prod_{i=1}^{n}\prod_{j=1}^{m(i)} \mathbb{P}(A_{i,j}) = \prod_{i=1}^{n} \mathbb{P}(\bigcap_{j=1}^{m(i)} A_{i,j}).$$

Therefore, by (D2.1.7) the $\mathcal{B}_i$'s are independent. □