

## Strong Law of Large Numbers

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### Theorem: WLLN, D2.4.1

Let  $X_1, X_2, \dots$  be i.i.d. (pairwise in fact suffice) with  $\mathbb{E}|X_1| < \infty$ . Then  $S_n/n \rightarrow \mu = \mathbb{E}X_1$  almost surely.

*Proof. Idea:* we assume  $X_1 \geq 0$  or otherwise we use  $X = X^+ - X^-$ . Then  $S_n$  and  $n$  are both increasing in  $n$ . Consider a subsequence, say  $k(n) = \lfloor \alpha^n \rfloor$  with  $\alpha > 1$  but close to 1. For the indices in between the subsequences, i.e., for  $k(n) \leq m \leq k(n+1)$ ,

$$\frac{S_{k(n)}}{k(n)} \frac{k(n)}{k(n+1)} = \frac{S_{k(n)}}{k(n+1)} \leq \frac{S_m}{m} \leq \frac{S_{k(n+1)}}{k(n)} = \frac{S_{k(n+1)}}{k(n+1)} \frac{k(n+1)}{k(n)}.$$

As  $n \rightarrow \infty$ ,  $k(n)/k(n+1) \rightarrow 1/\alpha$  and  $k(n+1)/k(n) \rightarrow \alpha$ . Therefore if we show convergence of the subsequence  $S_{k(n)}/k(n) \rightarrow \mu$ , then

$$\frac{\mu}{\alpha} \leq \liminf_{m \rightarrow \infty} \frac{S_m}{m} \leq \limsup_{m \rightarrow \infty} \frac{S_m}{m} \leq \alpha\mu,$$

and since  $\alpha$  is arbitrary, we are done.

*Proof of SLLN. Step 1.* We truncate as usual: let  $Y_n = X_n 1_{\{X_n \leq n\}}$  and let  $T_n = \sum_{i=1}^n Y_i$ . Then

$$\sum_{n \geq 1} \mathbb{P}(X_n \neq Y_n) = \sum_{n \geq 1} \mathbb{P}(X_n > n) = \sum_{n \geq 1} \mathbb{P}(X_1 > n) < \infty$$

since  $\mathbb{E}X_1 < \infty$ . Therefore, by B-C,  $\mathbb{P}(X_n \neq Y_n \text{ i.o.}) = 0$ . Since there are only finite number of different terms between  $S_n$  and  $T_n$ ,  $(S_n/n) - (T_n/n) \rightarrow 0$ . Therefore it suffices to show  $T_n/n \rightarrow \mu$  almost surely.

**Step 2.** We apply B-C to  $T_n/n$ . Using Chebyshev,

$$\mathbb{P}\left(\left|\frac{T_k - \mathbb{E}T_k}{k}\right| > \epsilon\right) \leq \frac{\text{var}(T_k)}{k^2 \epsilon^2} = \frac{1}{k^2 \epsilon^2} \sum_{i=1}^k \text{var}(Y_i).$$

But  $\text{var}(Y_i)$  may not  $\rightarrow 0$ . and then the terms on the RHS is bounded from below by some constant divided by  $k$ , not summable. Remedy:

**Step 3.** Apply step 2 to a subsequence  $k(n) = \lfloor \alpha^n \rfloor \geq \alpha^n/2$ . Then

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|T_{k(n)} - \mathbb{E}T_{k(n)}|}{k(n)} > \epsilon\right) &\leq \sum_{n=1}^{\infty} \frac{1}{\epsilon k(n)^2} \sum_{i=1}^{k(n)} \text{var}(Y_i) \\ &= \sum_{i=1}^{\infty} \epsilon^{-2} \text{var}(Y_i) \sum_{k(n) \geq i} \frac{1}{k(n)^2} \\ &\leq \sum_{i=1}^{\infty} 4\epsilon^{-2} \text{var}(Y_i) \sum_{k(n) \geq i} \alpha^{-2n} \\ &\leq \sum_{j=1}^{\infty} 4\epsilon^{-2} \text{var}(Y_j) \frac{1}{j^2} \frac{1}{1 - \alpha^{-2}} \\ &= \frac{4\epsilon^{-2}}{1 - \alpha^{-2}} \sum_{j=1}^{\infty} \frac{\text{var}(Y_j)}{j^2} \leq \frac{4\epsilon^{-2}}{1 - \alpha^{-2}} \sum_{j=1}^{\infty} \frac{\mathbb{E}Y_j^2}{j^2} \end{aligned} \quad (*)$$

Since

$$\mathbb{E}Y_j^2 = \int_0^{\infty} 2y \mathbb{P}(Y_j > y) dy \leq \int_0^j 2y \mathbb{P}(X_1 > y) dy,$$

the sum in (\*) becomes

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\mathbb{E}Y_j^2}{j^2} &= \sum_{j=1}^{\infty} j^{-2} \int_0^{\infty} 1_{\{Y < j\}} 2y \mathbb{P}(X_1 > y) \, dy \\ &= \int_0^{\infty} \underbrace{\left( \sum_{j>y} j^{-2} \right) 2y \mathbb{P}(X_1 > y)}_{\text{integrable}} \, dy \end{aligned} \quad (**)$$

Since  $\sum_{j>y} j^{-2} \approx y^{-1}$ , it can be shown that (D2.4.4)

$$\left( \sum_{j>y} j^{-2} \right) 2y \leq 4 \quad \text{for all } y.$$

Hence  $(**) \leq 4\mathbb{E}X_1 < \infty$ . Then (\*) and B-C says

$$\mathbb{P}\left(\frac{|T_{k(n)} - \mathbb{E}T_{k(n)}|}{k(n)} > \epsilon \text{ i.o.}\right) = 0 \quad \text{for all } \epsilon,$$

so

$$\frac{T_{k(n)} - \mathbb{E}T_{k(n)}}{k(n)} \rightarrow 0 \text{ a.s.} \quad \text{and} \quad \frac{T_{k(n)}}{k(n)} \text{ and } \frac{S_{k(n)}}{k(n)} \rightarrow \mu \text{ a.s.}$$

We have shown the a.s. convergence of a subsequence of  $S_{k(n)}/k(n)$ . By a remark made earlier we are done.  $\square$