MATH 574 Homework 2

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Problem 1

Let A be an $n \times n$ matrix over F.

(1) Show that A is nilpotent implies $A^n = 0$.

(2) Show that A is nilpotent iff its characteristic polynomial is x^n.

(3) (Extra credit) Show that if $F = \mathbb{C}$ then A is nilpotent if and only if the trace of $A^i = 0$ for all i. Show by example that this is not true if F has positive characteristic.

Proof. (1) Let m be the smallest integer with $A^m = 0$. If $m \leqslant n$ there is nothing to show. If $m > n$, then the min poly of A divides x^m. On the other hand, Cayley-Hamilton implies that the min poly of A has order at most n. This cannot happen, so $m \leqslant n$.

(2) If A is nilpotent, then all of its eigenvalues are 0, for otherwise if $\lambda > 0$ is an eigenvalue with eigenvector x, we gave $A^k x = \lambda^k x \neq 0$. Therefore, the characteristic polynomial has n roots at 0 and is x^n.

Conversely, if the char poly is x^n, then its min poly by Cayley-Hamilton can only be of form x^k for some $k \leqslant n$, and so regardless of what this k is, $A^k = 0$, and A is nilpotent.

(3) If A is nilpotent, then it only as zero as eigenvalues, so the trace raised to any power must still be 0.

Conversely, suppose A is not nilpotent. Let $\lambda_1, \ldots, \lambda_r$ be its nonzero eigenvalues, with dimensions of eigenspace $d_1, \ldots, d_r$. Then we have $Vd = 0$ where V is the Vandermonde matrix with first row $(\lambda_1, \ldots, \lambda_r)$ and $d = (d_1, \ldots, d_r)^T$. Since Vandermonde matrices are invertible (an easy result following induction), we must have $d_1, \ldots, d_r = 0$, contradiction. Hence A must be nilpotent. □

Problem 2

Prove that every 2×2 symmetric matrix over $\mathbb{R}$ is diagonalizable.

Proof. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Its char poly is

$$\det(\lambda I - A) = (\lambda - a)(\lambda - c) - b^2 = \lambda^2 - (a + c)\lambda + (ac - b^2).$$

The discriminant is

$$(a+c)^2 - 4(ac-b^2) = (a-c)^2 + 4b^2 \geqslant 0$$

with equality if and only if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$. It is clear that when this is not the case, the char poly has two distinct roots as eigenvalues of A, so A is diagonalizable. When A is a scalar multiple of I, it is still clearly diagonalizable as I is. □

Problem 3

Let J be the $n \times n$ matrix with all entries equal to 1. Find the minimal and characteristic polynomials of J. Is J diagonalizable?

Solution. Note that $J^2 = nJ$, implying that $J^2 - nJ = J(J - nI) = 0$. This is the min poly of J as it is monic and no degree 1 function on J kills it. Since the min poly $x^2 - nx$ has roots 0 and n, these are the only eigenvalues of J. Also, since the trace of A is n, we know the multiplicity of 0 is $n-1$. That is, the char poly of J is $x^{n-1}(x-n)$. To diagonalize J, one sufficient condition is that the characteristic $> n$ or $= 0$. If so, we define a matrix D by diagonal$(n, 0, \ldots, 0)$ and P by

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

namely the first column $= (1, \ldots, 1)$, and all other j^{th} column $= e_1 - e_j$. It is immediate that the first column is an eigenvector of J with eigenvalue A, and the other columns are independent eigenvectors with eigenvalues 0. Therefore, A under such assumption is diagonalizable.

Problem 4

Let J be the $n \times n$ matrix with all entries 0 except for the $(i, i+1)$ entries and assume they are all nonzero. Compute the min and char poly of J.

Solution. It is immediately clear that the char poly of J is $(-1)^n \lambda^n$. By Cayley-Hamilton, this means the min poly is of form λ^k, $k \leqslant n$. Note that J^2 is the matrix with all entries zero except $(i, i+2)$, J^3 except $(i, i+3)$, and so on. Going all the way up, J^{n-1} still has a nonzero entry $(1, n)$, and finally J^n is the min poly we seek.

Problem 5

Let n be odd and let K be a skew symmetric $n \times n$ matrix. Prove that $\det(K) = 0$.

Proof. If the characteristic of K is not 2, then $\det(K) = \det(K^T)$ implies $\det(K) = \det(-K)$, so $(-1)^n \det(K) = -\det(K) = \det(K)$, implying $\det(K) = 0$.

If the characteristic of K is 2, we brute force expand $\det(K)$ using summation $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} K_{i,\sigma(i)}$. Since K

has diagonal terms zero, any σ involving one fixed point will evaluate to 0. On the other hand, all permutations with inverses can be paired with their inverses. Since the sign of a permutation is the same as its inverse, and since

$$\prod_{i=1}^{n} K_{i,\sigma(i)} = (-1)^n \prod_{i=1}^{n} K_{i,\sigma^{-1}(i)} = \prod_{i=1}^{n} K_{i,\sigma^{-1}(i)}$$

in a char 2 field, we have $\operatorname{sgn}(\sigma)\prod_{i=1}^{n} K_{i,\sigma(i)} + \operatorname{sgn}(\sigma^{-1})\prod_{i=1}^{n} K_{i,\sigma^{-1}(i)}$ = either $0+0$ or $1+1$, both of which $= 0$. Therefore, the determinant of K is 0. □

Problem 6

Give examples of two matrices with the same char poly and min poly but are not similar.

Solution. Using problem 4, we consider A with $A_{1,2} = 1$ and B with $B_{1,2} = B_{3,4} = 1$, and zero on all other entries. Both have char poly x^4 and min poly x^2. However, A and B consist of different Jordan blocks, so they are not similar.