MATH 574 Homework 3

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Problem 1

Let $A \in M_n(\mathbb{C})$.

(1) Show that if $\lim_m A_m = B$ then $\lim_m U^{-1}A_mU = U^{-1}BU$.

(2) Show that $\lim_m A^m = 0$ iff all eigenvalues of A have absolute value less than 1.

(3) Show that $\{A^m : m \geqslant 1\}$ is bounded iff all aigenvalues has absolute value at most 1 and every Jordan block of A corresponding to an eigenvalue of absolute value 1 has size 1.

(4) If $A = SJS^{-1}$

Proof. (1) Once U is fixed, $A \mapsto U^{-1}AU$ is continuous. Therefore it preserves limits, and in particular $U^{-1}A_mU \to U^{-1}BU$.

(2) If A has an eigenvalue λ with $|\lambda| \geqslant 1$, let v be the corresponding eigenvector and WLOG assume $\|v\| = 1$. Then $\|A^kv\| \geqslant 1$ for all k, and continuity implies it is impossible that $A^k \to 0$, since $\|0v\| = 0$.

Conversely, let $\|\cdot\|$ be any vector norm. By assumption, all eigenvalues have moduli strictly less than 1, so their maximum satisfies this properly too. In particular, $\sup_{\|x\|=1} \|Ax\| < 1$, i.e., the subordinate norm of A < 1. Taking powesr, we see with respect to this norm, $\|A^m\| \to 0$. This implies $A^m \to 0$.

(3) Let J be a Jordan form such that $A = SJS^{-1}$. Then $A^k = SJ^kS^{-1}$. In order for A^k to be bounded, it is necessary and sufficient that J^k is bounded. For each block $J(\lambda_k)$, if $|\lambda_k| < 1$, the powers are clearly bounded. When $|\lambda_k| > 1$, the powers of this block are clearly unbounded. If $|\lambda_k| = 1$, the powers are bounded if (and only if) the block is 1×1, since for larger blocks (e.g. blocks of size 2), we would have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

which becomes unbounded as $k \to \infty$. We have therefore shown the claim. □

Problem 2

Problem 3

Let $A \in M_n(\mathbb{C})$.

(1) Show that e^A exists.

(2) Show that if $AB = BA$ then $\exp(A+B) = \exp(A)\exp(B)$.

(3) Show that it $AB \neq BA$, then (2) may fail.

(4) Show that $\exp(A)$ is invertible.

(5) Show that $\det(\exp(A)) = \exp(\operatorname{tr}(A))$.

Proof. (1) Since matrix norms are submultiplicative,

$$\left\| \sum_{m=0}^{\infty} \frac{A^m}{m!} \right\| \leqslant \sum_{m=0}^{\infty} \frac{\|A^m\|}{m!} \leqslant \sum_{m=0}^{\infty} \frac{\|A\|^m}{m!} = \exp(\|A\|) < \infty.$$

(2) If $AB = BA$, then $A^i B^j$ can be re-ordered arbitrarily without changing the value. In particular, binomial theorem gives

$$\exp(A+B) = \sum_{m=0}^{\infty} \frac{(A+B)^m}{m!} = \sum_{m=0}^{\infty} (n!)^{-1} \sum_{k=0}^{m} \binom{m}{k} A^k B^{m-k} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{A^k}{k!} \frac{B^{m-k}}{(m-k)!} = \sum_{i \geqslant 0} \frac{A^i}{i!} \sum_{j \geqslant 0} \frac{B^j}{j!} = \exp(A)\exp(B).$$

(3) Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so that $AB = 0 \neq BA = A$. Then it is clear that $A^k = 0$ for any $k \geqslant 2$, and $B^k = B$ for all $k \neq 0$. But then

$$\exp(A)\exp(B) = (I + A)(I + eB) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$$

whereas $A + B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, so again $(A+B)^k = (A+B)$ for all $k \neq 0$, and we have

$$\exp(A+B) = I + \begin{bmatrix} 0 & 0 \\ e & e \end{bmatrix} \neq \exp(A)\exp(B).$$

(4) Clearly $\exp(0) = I$, and also $A(-A) = (-A)A$, so by (b) $\exp(A + (-A)) = I = \exp(A)\exp(-A) = \exp(-A)\exp(A)$. Therefore $\exp(-A)$ is the inverse of $\exp(A)$.

(5) We transform A into its Jordan canonical form using matrix S (i.e., $A \mapsto SAS^{-1}$). Then

$$\det(\exp(A)) = \det\exp(S^{-1}JS) = \det(S^{-1}\exp(J)S) = \det(S^{-1})\det(\exp(J))\det(S) = \det(\exp(J)).$$

It is clear that the determinant of $\exp(J)$ is just the product of $\exp(J_{i,i})$, where $J_{i,i}$ is a diagonal entry of J. But then this means $\det(\exp(A)) = \det(\exp(J)) = \exp(\operatorname{tr}(J)) = \exp(\operatorname{tr}(A))$. □

Problem 4

Let A, B be Hermitian. Show that AB is Hermitian iff $AB = BA$.

Proof. If AB is Hermitian, then $(AB)^* = B^* A^* = BA$, since A, B are Hermitian.

Conversely, if $AB = BA$, then $(AB)^* = B^* A^* = BA = AB$, again, since A, B are Hermitian. □

Problem 5

Let $\rho(A)$ be the maximum of the absolute values of the eigenvalues of A, i.e., the special radius of A. If $AB = BA$, show $\rho(A + B) \leqslant \rho(A) + \rho(B)$.

Proof. Let another norm $\|\cdot\|$ be given. From lecture we know $\rho(A) = \lim_n \|A^n\|^{1/n}$ and $\|A^n\|^{1/n} \leqslant \|A\|^{n(1/n)} = \|A\|$. Let $\epsilon > 0$. Let $a \in [\rho(A), \rho(A) + \epsilon)$ and similarly let $b \in [\rho(B), \rho(B) + \epsilon)$. Then there exists N sufficiently large such that if $n \geqslant N$, $\|A^n\|^{1/n} < a$ and $\|B^n\|^{1/n} < b$. Therefore, it is natural for us to bound $\|A^n\|^{1/n}$ by a for large n and $\|A\|$ for small n. In particular, if $n > 2N$,

$$\begin{aligned}
\|(A+B)^n\| &= \left\| \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \right\| \leqslant \sum_{k=0}^{n} \binom{n}{k} \|A^k\| \|B^{n-k}\| = \sum_{k=0}^{N-1} + \sum_{k=N}^{n-N} + \sum_{k=n-N+1}^{n} \\
&\leqslant \sum_{k=0}^{N-1} \binom{n}{k} \|A\|^k b^{n-k} + \sum_{k=N}^{n-N} \binom{n}{k} a^k b^{n-k} + \sum_{k=n-N+1}^{n} \binom{n}{k} a^k \|B\|^{n-k} \\
&= \sum_{k=0}^{N-1} \binom{n}{k} a^k b^{n-k} \left(\frac{\|A\|}{a}\right)^k + \sum_{k=N}^{n-N} \binom{n}{k} a^k b^{n-k} + \sum_{k=n-N+1}^{n} \binom{n}{k} a^k b^{n-k} \left(\frac{\|B\|}{b}\right)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \max(\cdot) =: C(a+b)^n.
\end{aligned}$$

where the maximum is taken over N terms of expressions of form $(\|A\|/a)^k, (\|B\|/b)^{n-k}$, and 1. Taking limits, and sending $a, b \to \rho(A), \rho(B)$, respectively,

$$\lim_{n\to\infty} \|(A+B)^n\|^{1/n} \leqslant \lim_{n\to\infty} C^{1/n}(a+b) = a + b$$

for all $a \geqslant \rho(A), b \geqslant \rho(B)$, so indeed $\rho(A+B) \leqslant \rho(A) + \rho(B)$. □

Problem 6

Let $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$. Let $D = A \otimes B$.

(1) Show that the eigenvalues of D are precisely the products of an eigenvalue of A with an eigenvalue of B.

(2) Show that the char poly of D is the product of the char polys of A and B.

(3) Show that $\operatorname{tr}(D) = \operatorname{tr}(A)\operatorname{tr}(B)$.

Proof. (1) If $Av = \lambda v$ and $Bw = \nu w$, then $(A \otimes B)(v \otimes w) = (Av) \otimes (Bw) = \lambda\nu(v \otimes w)$.

(2) I do not think this claim holds. For example consider $A = [1]$ and $B = [2]$. Clearly the char polys are

$(x-1)$ and $(\lambda-2)$. Then $A \otimes B = [2]$, and the char poly is still $(\lambda-2) \neq (\lambda-1)(\lambda-2)$.

(3) We express D in terms of block matrices. The diagonal entries of D are entirely contained in the diagonal blocks $a_{i,i}B$:

$$D = \begin{bmatrix} [a_{1,1}B] & & & \\ & [a_{2,2}B] & & \\ & & \ddots & \\ & & & [a_{m,m}B] \end{bmatrix}$$

The block $a_{i,i}B$ has trace $a_{i,i}\text{tr}(B)$. Summing over all $i \in [1, m]$, we see the trace of D is nothing but $\text{tr}(A)\text{tr}(B)$. □