574 Homework #1 – Due Tuesday, September 20, 2022

(1) Let V, W be finite dimensional vector spaces over a field F and $T : V \to W$ a linear transformation.
 (a) Show that $\ker(T) = 0$ if and only if T maps any linearly independent subset of V to a linearly independent subset of W.
 (b) Show that $T(V) = W$ if and only if T takes any spanning subset of V to a spanning a subset of W.
 (c) Show that T is an isomorphism of V and W if and only if T maps every basis of V to a basis for W.

(2) Let $T : M_n(F) \to M_n(F)$ be the map $T(A) = A - A^\top$ (the transpose of A).
 (a) Describe $\ker(T)$ and compute $\dim \ker(T)$.
 (b) Describe the image of T and compute its dimension.
 (c) Show that if the characteristic of F is not 2, then $\ker(T) \cap \operatorname{im}(T) = 0$.

(3) Let F be a finite field with q elements (one can show that q is always a power of a prime).
 (a) Show that if V is a d-dimensional vector space over F, then $|V| = q^d$.
 (b) Recall that $\mathrm{GL}_n(F)$ is the group of $n \times n$ invertible matrices over F. Prove that

$$|\mathrm{GL}_n(F)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

 Hint: What can you say about the rows (or columns of an element $A \in \mathrm{GL}_n(F)$).
 (c) Let $\mathrm{SL}_n(F)$ denote the subgroup of $\mathrm{GL}_n(F)$ consisting of all matrices of determinant 1. Determine $|GL_n(F)|/|SL_n(F)|$.

(4) Let A, B be $m \times m$ matrices over a field F. A and B are similar if there exists an invertible matrix U so that $U^{-1}AU = B$.
 (a) Show that if A is invertible, then AB and BA are similar.
 (b) Show by example that this need not be the case if A and B are not invertible.
 (c) Prove that $\det(AB) = \det(BA)$.
 (d) Prove that AB and AB have the same trace (the trace of a matrix is the sum of its diagonal entries).
 (e) Prove that if x is a variable, then $\det(xI - AB) = \det(xI - BA)$.

(5) Let $T : V \to V$ be a linear a transformation. Assume that $T^2 = T$. (Such an operator is called an idempotent). Prove that every vector $v \in V$ can be written uniquely as $v_1 + v_2$ where $v_1 \in \operatorname{im}(T)$ and $v_2 \in \ker(T)$.