MATH 574 Homework 4

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Problem 1

Problem 2

Proof. (1) By assumption, $A \in M_n(\mathbb{C})$ satisfies $AA^* = A^*A$, i.e., A is normal, so it is unitarily similar to a diagonal matrix, making it diagonalizable. To show all the eigenvalues are complex, we note that if $Av = \lambda v$ then from $A + A^* = 0$ we obtain

$$0 = v^*(A + A^*)v = v^*Av + v^*A^*v = \lambda v^*v + \overline{\lambda} v^*v = 2\mathfrak{Re}(\lambda)v^*v.$$

By assumption $\|v\| \neq 0$, so λ is purely imaginary.

(2) Since A and B are similar, they share the same eigenvalues. Therefore are orthogonally similar to the same matrix of eigenvalues, and in particular, they themselves are orthogonally similar too.

(3) Treated as element in $M_n(\mathbb{R})$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is not diagonalizable: its characteristic polynomial is $\lambda^2 + 1$, which has no real roots. □

Problem 3

Proof. [Since this is titled Homework 4, I consulted our textbook when attempting this one.]

Since A is Hermitian, B clearly is; also, since $x^T Ax \geqslant 0$ for all x, the components corresponding to those included in B are also positive, and this establishes positive definiteness.

Regarding eigenvalues, both parts of this problem are consequences of Theorem 4.3.17 from the book. Given B, we let P be the permutation matrix such that

$$PAP^T = \begin{bmatrix} B & * \\ * & * \end{bmatrix}.$$

In particular, the theorem states that the maximal eigenvalue of B is squeezed between $\lambda_{n-d}(PAP^T)$ and $\lambda_{n-d+1}(PAP^T)$, so it is in particular bounded by $\lambda_1(PAP^T)$. Since P is a permutation matrix, PAP^T and A share the same eigenvalues, and we are done. □

Problem 4

Proof. (1) If A has rank one then its column space is one-dimensional. That is, each column is of form $c_i v$ where $c_i \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Then $A = v(c_1, c_2, ..., c_n)^T$. The converse is trivial.

To show v is an eigenvector of A, we simply note that

$$Av = vw^T v = (w^T v)v$$

so v is an eigenvector with eigenvalue $w^T v$.

(2) If A is in addition nonnegative, then all its columns are nonnegative and are scalar multiples of each other. Therefore can simply pick v to be the first column and scale $u = (c_1, c_2, ..., c_n)$ by $c_j = A_{1,j}/A_{1,1}$.

(3) Rank-nullity theorem suggests that $\dim\ker(A) = n-1$. Since tr(A) is also the sum of eigenvalues, the char poly of A is simply $x^{n-1}(x - \text{tr}(A))$. It follows that if tr$(A) \neq 0$ then we have two eigenspaces, one corresponding to tr(A) with dimension 1, the other to 0 with dimension $n-1$, so in this case A is diagonalizable. On the other hand if tr$(A) = 0$ then we only have one eigenspace with dimension $n-1$, and we are unable to find n linearly independent eigenvectors, so A is not diagonalizable. This completes the proof. □

Problem 5

Proof. (1) Define $T_1(X) = AX$ and $T_2(X) = XB$. It is clear that

$$T_2T_1(X) = (AX)B = AXB = A(XB) = T_1T_2(X),$$

so T_1, T_2 commute. Therefore they can be simultaneously diagonalized by a matrix, say Q. Then $Q^*(A - B)Q$ is also upper triangular with diagonal entries as differences between eigenvalues of T_1 and T_2, i.e., numbers of form $a - b$, where a is an eigenvalue of A and b an eigenvalue of B.

(2) Let a, b be given. Let u, v be such that $Au = au$ and $vB = bB$. Then

$$T(uv^*) = Auv^* - uv^*B = auv^* - buv^* = (a-b)uv^*,$$

showing that $(a - b)$ is indeed an eigenvalue!

(3) If A and B share some eigenvalue λ, then letting u, v be A's and B's corresponding eigenvector eigenvectors, respectively, using (2) we see $T(uv^*) = 0$. Thus T is not invertible.

Conversely, if T is invertible and $uv^* \neq 0$ then $T(uv^*) \neq 0$. Letting u traverse through all eigenvectors (one for each eigenvalue) of A and v for B, we see that A and B cannot share any common eigenvector.

(4) If $A = B$, $\ker(T) = \{X : AX = XA\}$, i.e., its centralizer. By a HW3 question this space has dimension at least n, and we are done.

□

Problem 6

Proof. From our previous homework we know that B exists (and is finite) if and only if (i) all eigenvalues of A have absolute value at most 1 and (ii) every Jordan block of A corresponding to an eigenvalue ±1 has size 1. For this question to even make sense, let us assume these a priori, and write $A = SJS^{-1}$.

Since S, S^{-1} are continuous transformations,

$$B = \lim_{n\to\infty} A^n = S[\lim_{n\to\infty} J^n]S^{-1},$$

where $\lim_{n\to\infty} J^n = 0$ except on diagonal entries where $J_{i,i} = 1$. Squaring such a matrix changes nothing, and indeed

$$B^2 = S[\lim_{n\to\infty} J^n]S^{-1}S[\lim_{n\to\infty} J^n]S^{-1} = B.$$

Next up, note that the rank of J is the number of nonzero entries (or more precisely, the number of ones on diagonal), and this corresponds to the A having 1 as an eigenvalue with multiplicity $\text{rank}(B)$. □