

Unitary Linear Transformation

A linear transformation $T : V \rightarrow V$ on an inner product space is **unitary** if and only if $(Tv, Tw) = (v, w)$. (If over the reals, this is the same as orthogonal.)

Theorem

If $T : V \rightarrow V$ is unitary, then there exists an orthonormal basis of eigenvectors of T .

Proof. Let v be an eigenvector for T . WLOG assume $\|v\| = 1$. Let $W = v^\perp$ the orthogonal complement. By Gram-Schmidt, any vector can be decomposed into a combination of v and something in W .

Claim: $TW = W$. If $w \in W$, then $(Tw, Tv) = (Tw, \lambda v) = \bar{\lambda}(Tw, v) = 0$. Since $\bar{\lambda} \neq 0$, we have $(Tw, v) = 0$.

By induction, T has an orthonormal basis of eigenvectors v_1, v_2, \dots, v_n . Since T is unitary, $|\lambda| = 1$ for all eigenvalues by the fact that $(Tv, Tv) = |\lambda|^2(v, v)$. \square

Hermitian Operators

We say T is **Hermitian** if $(Tv, w) = (v, Tw)$ for all v, w .

The same theorem above holds verbatim.

Proof. We first note that every eigenvalue of a Hermitian T is real: if $Tv = \lambda v$ then

$$\lambda(v, v) = (Tv, v) = (v, Tv) = \bar{\lambda}(v, v).$$

Next, we note that eigenvectors corresponding to different eigenvalues are orthogonal: if $Tv = \lambda v$, $Tw = \mu w$, and $\lambda \neq \mu$, then

$$\lambda(v, w) = (Tv, w) = (v, Tw) = \mu(v, w).$$

We now show T is diagonalizable. If not, there exist vectors v_1, v_2 , with $Tv_1 = v_1$, and $Tv_2 = \lambda v_2 + v_1$.

Replacing T by $T - \lambda I$, we have $(T - \lambda I)v_1 = 0$, and $(T - \lambda I)v_2 = v_1$. Then, since the sum of two Hermitian matrices is still Hermitian, if T is, then so is $T - \lambda I$, so

$$0 = ((T - \lambda I)v_1, v_2) = (v_1, (T - \lambda I)v_2) = (v_1, v_1) \neq 0.$$

Therefore, T is diagonalizable, and we let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the distinct eigenvalues and $V = V_1 \oplus \dots \oplus V_r$. Furthermore, by the second fact we showed, each V_i is orthogonal with each other. We simply pick the orthogonal bases of each eigenspace. The union is therefore an orthogonal basis of V . \square

Normal Operators

Again, let V be a vector space over \mathbb{C} of dimension n . Let $T : V \rightarrow V$ and define $T^* : V \rightarrow V$ by $(Tv, w) = (v, T^*w)$.

[We claim there is one unique T^* . To do this, we let e_1, \dots, e_n be a basis of V and require $(e_i, T^*e_j) = (Te_i, e_j)$.]

Using this notation, T is Hermitian iff $T = T^*$, and T is **normal** if $TT^* = T^*T$.

Example. If v_1, v_2, \dots, v_n are orthonormal with $Tv_i = \lambda_i v_i$, then $T^*v_i = \bar{\lambda}_i v_i$, and $TT^* = T^*T$.

Like above, T is normal iff T has an orthonormal basis of eigenvectors.