Beginning of Oct. 19, 2022

Let H be Hermitian. Then H has an orthonormal basis of eigenvectors, and all eigenvalues are real. We say H is **positive definite** if all eigenvalues are positive, and **positive semidefinite** if all eigenvalues are nonnegative. Note that H defines a sesquilinear form on V, the sapce of column vectors, by

$$(v,w) = v^T H\overline{w} \qquad (v,v) = v^T H\overline{v}.$$

Therefore, H is positive definite iff the sesquilinear form forms an inner product, i.e., $v^T H\overline{v} > 0$ for all $v \neq 0$.

Proposition

If $A \in M_n(\mathbb{C})$, then AA^* is PSD Hermitian.

Proof. That AA^* is Hermitian is trivial. Note that since $(Av,w) = (v,A^*w)$ by definition, we have

$$(AA^*v,v) = (A^*v,A^*v) \geqslant 0.$$

If v is an eigenvector for A^*, we see the corresponding λ of A^* is nonnegative. □

The above claim still holds if A is any rectangular matrix. Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_s > 0$ be the positive eigenvalues of AA^*. Let $\mu_i = \sqrt{\lambda_i}$. The μ_i's are called the **singular values** of A.

Theorem: Singular Value Decomposition

Let $A \in M_n(\mathbb{C})$. Then there exist U,V unitary with $UAV = D$, diagonal $(\mu_1,\ldots,\mu_s,0,\ldots)$ where μ_i are the singular values of A, so $A = U^*DV^*$.

Proof. If U,V are unitary, and $B = UAV^*$, then

$$BB^* = (UAV^*)(V^{**}A^*U^*) = UAA^*U^* = UAA^*U^*.$$

Therefore, AA^* is similar to BB^*. On the other hand,

$$B^*B = (VA^*U^*)(UAV^*) = VA^*AV^*$$

so B^*B is also similar to A^*A, which is similar to AA^*, and they are both similar to diag $(\lambda_1,\ldots,\lambda_s,0,\ldots)$. Following this reasoning, we can pick U,V such that

$$U(AA^*)U^* = V(A^*A)V^* = D.$$

Therefore $B = UAV^*$ is normal. Since B is normal, there exists a unitary matrix X with $XBX^* = (XU)A(U * X^*)$ equal to diag $(\rho_1,\ldots,\rho_n)$ with $|\rho_1| \geqslant |\rho_2| \geqslant \ldots$. Since each $z \in \mathbb{C}$ can be written as $\alpha\beta$ with $|\beta| = 1$ and $\alpha \in \mathbb{R}$, we can left multiply by a matrix diag (β_i^{-1}) and obtain $Y_1AY_2^* =$ diag $(\gamma_1,\ldots,\gamma_n)$ with $\gamma_1 \geqslant \gamma_2 \geqslant \ldots$ real and nonincreasing. Then $(Y_1AY_2^*)(Y_1AY_2^*)^*$ is PD with eigenvectors as that of AA^*, and we are done. □