Beginning of Oct. 5, 2022

Unitary Matrices

It is easy to check that if $A = A^*$, then $(\lambda A)^* = \overline{\lambda} A^* = \overline{\lambda} A = \lambda A$ iff $\lambda \in \mathbb{R}$. That is, Hermitian matrices are a vector space over $\mathbb{R}$, but not $\mathbb{C}$.
By the same token, the set of skew symmetric matrices are a vector space over $\mathbb{R}$ only.
Let $\mathrm{H}_n, \mathrm{SH}_n$ be the two sets, respectively.
Note that $\mathrm{SH}_n \cap \mathrm{H}_n = \{0\}$, and $M_n(\mathbb{C}) = \mathrm{SH}_n \oplus \mathrm{H}_n$ over the reals:

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}.$$

Furthermore, $\mathrm{SH}_n = \mathrm{H}_n$, so $\dim \mathrm{SH}_n = \dim \mathrm{H}_n$.
We say $A \in M_n(\mathbb{C})$ is **unitary** if $A^* = A^{-1}$. We say A is **normal** if $AA^* = A^*A$. In particular, Hermitian and unitary matrices are normal.

Proposition

If A, B are Hermitian, AB is Hermitian iff $AB = BA$.

Bilinear forms over a vector space *V*

Let $B : V \times V \to F$ be bilinear. We find its Gram matrix via

$$G = \{B(v_i, v_j)\}$$

where each v_i is a basis vector. B is called **symmetric** if $B(v, w) = B(w, v)$, and this is equivalent to requiring G to be symmetric. Similarly, B is **alternating** if $B(v, v) = 0$ for all 0 (taking care of char 2; more generally for char non-2 fields, $B(v, w) = -B(w, v)$). This corresponds to G being skew symmetric.
Change of basis gives a new Gram matrix of form UGU^T, which is **congruent** to G. Conversely, if G is an $n \times n$ matrix with V being its column vectors, then we can define

$$B(v, w) := v^T G w$$

and recover the gram matrix G with standard basis of V. If $G = I$, the bilinear form becomes the inner product.

In $\mathbb{C}$, we have the inner product defined as

$$(\alpha_1, \ldots, \alpha_n) \cdot (\beta_1, \ldots, \beta_n) := \sum_{i=1}^{n} \alpha_i \overline{\beta}_i.$$

This is an example of a **sesquilinear form** on $V/\mathbb{C}$, (u, w):

- $\mathbb{C}$-linear in the first coordinate, and
- additive in the second coordinate.

Namely, $(v, \lambda w) = \overline{\lambda}(v, w)$.

Proposition

If V is an inner product space with dimension n over $\mathbb{C}$ or $\mathbb{R}$, then there exists an orthonormal basis $v_1, ..., v_n$. Easy proof by Gram-Schmidt.

The above claim implies G, a Gram matrix, represents an inner product iff $G = UU^*$ for some invertible U.

Proposition

U is unitary iff $(Uv, Uw) = (v, w)$ for all v, w.

Proof. It suffices to show $(Ue_i, Ue_j) = (v_i, v_j)$. □