Beginning of Aug 24, 2022

Proposition: Exchange Lemma

Suppose V is a finite-dimensional vector space. Suppose $w_1, \ldots, w_m$ span V and suppose $u_1, \ldots, u_n$ form a basis of V. Then there exists w_j sch that if we replace some u_i by w_j, $u_1, \ldots, u_n$ still form a basis. Also, $sn \leqslant m$.

Proof. WLOG assume each $w_i \neq 0$. Assume $w_1 = \sum_{i=1}^{n} \alpha_i u_i$ where α_i's are the coefficients. WLOG assume $\alpha_1 \neq 0$. Then u_1 can be written as a linear combination of $w_1, u_2, \ldots, u_n$. Hence $w_1, u_2, \ldots, u_n$ form a basis. □

Definition: Rank

If R is a ring and M a free R-module, we define the **rank** of M, $\operatorname{rank}(M)$, to be the size of its basis.
Fact: if R is a commutative ring and M is finitely generated, then $\operatorname{rank}(M)$ is well-defined.
In the infinite dimensional space, it is not hard to show bases exist using Zorn's lemma (and AC)

Homomorphisms

Let G, H be groups. Let R, S be rings.

(1) A **group homomorphism** is a map $f : G \to H$ such that $f(g_1 g_2) = f(g_1) f(g_2)$ for all $g_1, g_2 \in G$.

- This definition implies $f(e_G) = e_H$ and so $f(g^{-1}) = f(g)^{-1}$.
- Note this implies $f(e_G) = e_H$ and so $f(g^{-1}) = f(g)^{-1}$.
- Also, under such homomorphism, the image $f(G)$ is a subgroup of H.
- Finally, we define the **kernel** of f by

$$\ker(f) := \{g \in G : f(g) = e_H\}.$$

(2) A subgroup N of G is called **normal** if for all $g \in N$:

- $gN = Ng$,
- $gNg^{-1} = N$, and
- $N = \ker(\varphi)$ for some homomorphism $\varphi : G \to H$ for some H.

(3) If G is a group and N normal, we define the **quotient group** G/N to be

$$G/N := \{gN : g \in G\},$$

i.e., the set of all (left) **cosets** of H. Clearly, the operation is defined by $(g_1 N)(g_2 N) = (g_1 g_2) N$ with $(g_1 N)^{-1} = g_1^{-1} N$ and identity $e_G N = N$.

- This naturally leads to the group homomorphism $\pi : G \to G/N$ defined by $g \mapsto gN$ with kernel N.

(4) **Lagrange's theorem**: G is finite and N is a normal subgroup of G then $|N|$ divides $|G|$.

(5) A **ring homomorphism** is a map $\varphi : R \to S$ satisfying

- $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$,
- $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$, and
- $\varphi(1_R) = 1_S$ (and automatically $\varphi(0_R) = 0_S$).

(6) A subset $I \subset R$ is called a **left ideal** if I is an additive subgroup of R with $rm \in i$ for all $r \in R, m \in I$. A **right ideal** is similarly defined, and so is a **2-sided ideal**.

(7) Let I be a 2-sided ideal. We construct the **quotient ring** R/I by $\{r + I : r \in R\}$, with $(r+I)(s+I) = rs + I$.

(8) Let M_1, M_2 be left R-modules. A map $f : M_1 \to M_2$ is called an ***R*-module homomorphism** if

- $f(m_1 + m_2) = f(m_1) + f(m_2)$ and
- $f(rm_1) = rf(m_1)$

for $r \in R$ and $m_1, m_2 \in M$. The kernel of such f is a ***R*-submodule**.

Linear Transformations

Let F be a field; let V, W be vector spaces over F. If $T : V \to W$ is a module homoomorphism, we call T a **linear transformation**.

Proposition: Rank-Nullity Theorem

Suppose V is a finite dimensional vector space and W a subspace. Then $\dim V = \dim W + \dim V/W$.
This can be reformulated as follows: let T be a linear transformation from V to U where $\dim(V) < \infty$. Then $\dim V = \dim T(V) + \dim \ker(T)$.

Proof Sketch. Let $a_1, \ldots, a_m$ be a basis for W and let $a_1, \ldots, a_m, b_1, \ldots, b_{n-m}$ be a basis for V. It remains to notice that $b_1 + W, \ldots, b_{n-m} + W$ forms a basis for V/W. □

From this we see there exists a canonical **isomorphism** (bi-directional homomorphism) from $T(V)$ to V/W:

$$\varphi : V/W \to T(V) \qquad \text{with} \qquad \varphi(v + W) := Tv.$$

Direct Sum, Quotients, and Tensor Products

(1) Let V_1, V_2 be vector fields over F. We define the **direct sum** to be

$$V_1 \oplus V_2 := \{(v_1, v_2) : v_i \in V_i\}.$$

In particular, $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

(2) We define a relation $\otimes$ by $\alpha(v_1 \otimes v_2) = \alpha v_1 \otimes v_2 = v_1 \otimes \alpha v_2$. More generally,

$$\sum_i \alpha_i u_i \otimes \sum_j \beta_j v_j = \sum_{i,j} (\alpha_i \beta_j)(u_i \otimes v_j).$$

Therefore, if V_1 has basis $u_1, ..., u_m$ and V_2 has basis $w_1, ..., w_n$, $u_i \otimes w_j$ is a basis for $V_1 \otimes V_2$, the **tensor product**. In particular, $\dim(V_1 \otimes V_2) = \dim(V_1)\dim(V_2)$.