Beginning of Aug.29, 2022

Lemma

Let R be a ring. Let V be a free R-module with a basis $v_1, \dots, v_d$. If M is an R-module and $x_1, \dots, x_d \in M$ (just random elements, not necessarily distinct or basis), then there exists a unique R-module homomorphism $T : V \to M$ with $T(v_i) = x_i$.

Proof. We fist show existence. Define

$$T(\sum_{i=1}^{d} r_i v_i) := \sum_{i=1}^{d} r_i x_i.$$

Since the v_i's form a basis, this representation is well-defined for arbitrary inputs. It is immediate that this transformation is linear and satisfies $T(v_i) = x_i$.

Now for uniqueness, suppose T_1, T_2 both satisfy the requirements. It follows that $(T_2 - T_1)(v_i) = 0$ for all i (note $T_2 - T_1$ is an R-module homomorphism. By linearity $T_2 - T_1 \equiv 0$, completing the proof of uniqueness. □

Let R be a commutative ring and S, T be linear transformations from M to N. Then it follows that $S + T$ and rT for $r \in R$ are also well-defined linear transformations. That is, $\text{Hom}_R(M, N) := \{T : M \to N : T \text{ is a } R\text{-module homomorphism}\}$ is another module / vector space.

In particular, if $\dim(M) = m$ and $\dim(N) = n$, then $\dim \text{Hom}_R(M, N) = mn$. (Think matrices!)

Let F be a field and V_1, V_2 vector spaces over F. Let $T_i : V_i \to W_i$ be linear transformations. Then $T \equiv T_1 \otimes T_2 : V_1 \otimes V_2 \to W_1 \otimes V_2$ defined by

$$(T_1 \otimes T_2)(v_1 \otimes v_2) := T_1 v_1 \otimes T_2 v_2$$

is a well-defined linear transformation.

Lemma

Let F be a field and V, W finite-dimensional vector fields over F. Let $T : V \to W$ be a linear transformation. Then there exists a basis $v_1, \dots, v_d$ of V and a basis $w_1, \dots, w_e$ of W such that $T(v_i) = w_i$ for $i \leqslant m$ and $T(v_i) = 0$ for $i > m$ for some m. We say m is the **rank** of T, or equivalently $\dim T(V)$.

Proof. We first choose a basis $v_{m+1}, \dots, v_d$ for $\ker(T)$. We extend it to a basis for V by adding $v_1, \dots, v_m$. Let $w_i = T(v_i)$ for $i \leqslant m$. We claim $w_1, \dots, w_m$ are linearly independent. Suppose $\sum_{i=1}^{m} a_i w_i = 0$. That is, $\sum_{i=1}^{m} a_i T(v_i) = T(\sum_{i=1}^{m} a_i v_i) = 0$. That is, the linear combination $\sum_{i=1}^{m} a_i w_i \in \ker(T)$ but $v_1, \dots, v_m \notin \ker(T)$. That means it must be 0, and consequently $a_i = 0$, and we are done. □

Corollary

Suppose $S, T : V \to W$ are linear transformations. Then there exist isomorphisms $\varphi : V \to V$ and $\gamma : W \to W$ with $T = \gamma S \varphi$ if and only if $\text{rank}(T) = \text{rank}(S)$.

The collection of all linear transformations from $V \to V$ where V is a vector space over field F will be denoted as either $\mathcal{L}(V)$, $\mathcal{L}(V,V)$, $\text{Hom}_F(V,V)$, or $\text{End}_F(V)$ (**endomorphisms**). We have $\dim \text{End}_F(V) = \dim(V)^2$.
Finally, we define the set of all invertible endomorphisms on V to be the **general linear** group (it is a group)

$$\text{GL}(V) := \{T \in \text{End}_F(V) : T \text{ invertible}\} = \{T \in \text{End}_F(V) : T \text{ bijective}\}$$

with $\text{id} : V \to V$ being the identity.

Lemma

If $\dim V = n < \infty$ and $T \in \text{End}_F(V)$ (or if $T : V \to W$ with $\dim W = \dim V$), then TFAE:

(1) T is invertible,

(2) T is injective,

(3) T is onto, and

(4) $\text{rank}(T) = n$.

Note that the claim fails if $\dim V = \infty$. Consider $V = \mathbb{R}[x]$. Let $T : V \to V$ be defined by $Tf(x) := f'(x)$. Every polynomial has a polynomial antiderivative so T is onto, but clearly it is not injective.

0.1 Matrices

Let R be a ring and $M_{m\times n}(R)$ denote the set of $m \times n$ matrices over R. Addition and scalar multiplication as defined normally and they form an Abelian group and an R-module. Matrix multiplication is also defined the standard way. As usual, matrix multiplication is distributive, associative, but not commutative. That is, $M_{n\times n}(R) = M_n(R)$ is a ring. The set of invariable matrices form a group, namely $\text{GL}_n(R)$.