Beginning of Sept. 12, 2022

Complex Matrices

- The automorphism $z \mapsto \overline{z}$ is a field automorphism on $\mathbb{C}$.
- We can extend such mapping to $M_{m\times n}(\mathbb{C})$ in the sense that $\overline{A+B} = \overline{A} + \overline{B}$.
- If $f(x) \in \mathbb{C}[x]$ is non-constant then f has a root. It factors into the form $\prod_{i=1}^{n}(x - x_i)$ with x_i not necessarily distinct the roots. In general, a field F on which non-constant polynomials have roots is called *algebraically closed*. (So $\mathbb{C}$ is algebraically closed.)
- The complex conjugate $A^* := \overline{A^T}$ of $A \in M_{m\times n}(\mathbb{C})$:
  - $\det(A^*) = \overline{\det(A)}$.
  - **Hermitian**: $A^* = A$. Since $(\alpha A)^* = \overline{\alpha}A^*$, given A Hermitian, $(\alpha A)^*$ is Hermitian iff $\alpha \in \mathbb{R}$. That is, the Hermitian matrices form a vector space over $\mathbb{R}$ but not $\mathbb{C}$.
  - **Skew Hermitian** matrices satisfy $A^* = -A$. Note that each skew Hermitian is i times a Hermitian and in fact this multiplication defines an isomorphism between the two groups of matrices.
  - **Unitary** matrices: $AA^* = I$. Hermitian real implies symmetric. Since $(AB)^* = B^*A^*$, $U_n(\mathbb{C})$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Also, $\det(A) = 1$ if A is unitary.

Characteristic Polynomials and Eigenvalues

Let $A \in M_n(F)$. The **characteristic polynomial** (char poly) of A is $\det(\lambda I - A)$ as a function of λ. By the definition of the matrix $\lambda I - A$, expanding its term shows that char poly is at most of order n [when a term is $(\lambda - a_{1,1})(\lambda - a_{2,2})\ldots(\lambda - a_{n,n})$] and when this happens, the other terms are at most of order $n-2$. Hence the char poly must be a monic polynomial, with highest order terms being $x^n - \mathrm{tr}(A)x^{n-1}$ and so on. The constant term is $\det(0\cdot I - A) = (-1)^n\det(A)$.

Definition: Eigenvalue

$\alpha \in F$ is an **eigenvalue** of $A \in M_n(F)$ iff $v \in \ker(\alpha I - A)$ or equivalently $Av = \alpha v$ for some nonzero v.

Let $T : V \to V$ be a linear transformation over F. For $\alpha \in F$,

$$V_{T,\alpha}\{v \in V : Tv = \alpha v\} = \ker(\alpha I - T)$$

is a subspace of V, and its dimension defines the **geometric multiplicity** of α as an eigenvalue of T. (α is an eigenvalue of T iff $\dim V_{T,\alpha} > 0$.)

We say $T : V \to V$ is **nilpotent** if some power of T is 0. From the definition we see this implies the only eigenvalue of T is 0.

Again, let $A \in M_n(F)$. The **minimal polynomial** (min poly) of A is the lowest degree monic (nonzero) polynomial $m_A(x)$ so $m_A(A) = 0$. Note that min poly always exists: $\{I, A, \ldots, A^{n^2}\}$ has n^2+1 elements so by pigeonhole two of them must be linearly dependent, so we obtain a n^2-degree polynomial candidate of min poly of A. By a generalized division algorithm, we see that m_A actually needs to divide the polynomial we proposed.