Beginning of Sept. 14, 2022

Definition: Diagonalizable LT

Let $T : V \to V$ be a linear transformation. Let $\dim V = n$. We say T is **diagonalizable** if there exists a basis of eigenvectors $v_1, \ldots, v_n$.

For $A \in M_n(F)$, A is diagonalizable if there exists an invertible matrix U with $U^{-1}AU$ = a diagonal matrix D, or equivalently $A = UDU^{-1}$.

Let $e_1, \ldots, e_n$ be the standard basis for the column vectors. Then $Ue_1, \ldots, Ue_n$ forms a new basis. Note that

$$AUe_1 = UDU^{-1}(Ue_1) = UDe_1 = \alpha(Ue_1)$$

where α_1 is the $(1, 1)$ entry of D. This holds similarly for other indices. showing that each Ue_i is an eigenvector with eigenvalue α_i.

For $A, B \in M_n(F)$, we say A, B are **similar** if there exists an invertible U with $U^{-1}AU = B$.

Matrices and Linear Transformation

Let V be n-dimensional and $v_1, \ldots, v_n$ be a basis. Let $T : V \to V$ be a linear transformation. Let A_T be the matrix of T w.r.t. $\{v_1, \ldots, v_n\}$. Then

$$Tv_j = \sum_{i=1}^{n} a_{i,j} v_i.$$

With a different basis, we obtain a matrix similar to A_T.

Properties of similar matrices: if A, B are similar with $B = U^{-1}AU$, then

- $\det(A) = \det(B)$.
- A and B have the same characteristic polynomial: $\det(xI - A) = \det(xI - B)$.
- $\operatorname{tr}(A) = \operatorname{tr}(B)$ (directly from above).
- $B^n = U^{-1}A^nU$. More generally, for any polynomial f, $f(B) = U^{-1}f(A)U$.
- A and B have the same minimal polynomial.
- $A \mapsto U^{-1}AU$ is a ring automorphism.

Conditions for Diagonalization

Recall that eigenvalues are precisely the roots of the char poly.

Suppose we know 0 is an eigenvalue. This means we can simply find all corresponding eigenvectors by solving $Ax = 0$.

More generally, for each eigenvalue α_i, the correpsonding eigenspace is simply $\ker(\alpha_i I - A)$.

Theorem

A is diagonalizable if and only if $\sum \dim V_i = \dim V$, where each V_i is an eigenspace.

Before proving the theorem, let us first note that nonzero vectors in different eigenspaces are linearly independent. This is done by induction and the base case is trivial. Suppose $V_1, \ldots, V_r, V_{r+1}$ are eigenspaces of eigenvalues $\lambda_1, \ldots, \lambda_{r+1}$ and the claim holds for $V_1, \ldots, V_r$. Suppose for contradiction that for $v_i \in V_i$, some combination $\beta_1 v_1 + \ldots + \beta_{r+1} v_{r+1} = 0$. Left multiplying by A gives

$$A(\sum_{i=1}^{r+1} \beta_i v_i) = \sum_{i=1}^{r+1} \lambda_i \beta_i v_i = 0$$

If $\lambda_{r+1} = 0$ then by induction we are done. Otherwise, multiplying $\beta_1 v_1 + \ldots + \beta_{r+1} v_{r+1}$ by λ_{r+1} and subtracting, we have

$$(\lambda_{r+1} - \lambda_1)\beta_1 r_1 + \ldots + (\lambda_{r+1} - \lambda_r)\beta_r v_r = 0.$$

The claim again holds from induction. With this claim, we now prove the theorem.

Proof. If $\sum \dim V_i = \dim V$, then if we pick a basis for V_i and take the union, the resulting set is linearly independent and therefore forms a basis of n elements. Hence A has a basis of eigenvectors and is therefore diagonalizable.

Conversely, if there exists a basis, they are linearly independent, so $\dim V_i \geqslant$ the number of basis vectors corresponding to α_i. Taking sum gives $\sum \dim V_i \geqslant \dim V$. The other direction is trivial, so $\sum \dim V_i = \dim V_i$. □

Corollary

If A has char poly $f_A(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$ where the α_i's are distinct, then A is diagonalizable.

Proof. The dimension of each α_i-eigenspace is at least 1, so $\sum \dim V_i \geqslant n$. So they must attain equality. □

If A is diagonal with α_i on its diagonal, the polynomial

$$m(A) = (A - \alpha_1 I)\ldots(A - \alpha_n I) = 0.$$

(Note that each term has one zero row/column and the position of the zero row in each term is different.)

Special case of the **Cayley-Hamilton** theorem:

- If A is diagonalizable then the min poly divides the char poly.
- If A has in addition n distinct eigenvalues then its min poly = its char poly.