Beginning of Sept.19, 2022

Theorem

Let $A \in M_n(F)$ with the characteristic polynomial $c_A(x)$ is $(x-\alpha_1)\ldots(x-\alpha_n)$, not necessarily distinct. Then there exists an invertible matrix U with $U^{-1}AU$ upper triangular.

Theorem

Let $T: V \to V$ be linear with $\dim V = n$. Suppose the char poly of T is $(x-\alpha_1)\ldots(x-\alpha_n)$, $\alpha_i \in F$. Then there exists a basis $v_1, \ldots, v_n$ with

$$\begin{aligned} Tv_1 &= \alpha_1 v_1 \\ Tv_2 &= \alpha_2 v_2 + \alpha_{1,2} v_1 \\ Tv_3 &= \alpha_3 v_3 + \alpha_{2,3} v_2 + \alpha_{1,3} v_1 \\ Tv_j &= \alpha_j v_j + \text{ lower terms.} \end{aligned}$$

Proof. Since α_1 is an eigenvalue, there exists nonzero v_1 with $Tv_1 = \alpha_1 v_1$. Now we work with the quotient space $V/\langle v_1 \rangle$. Define a mapping

$$\overline{T}: V/\langle v_1\rangle \to V/\langle v_1\rangle \qquad \text{by } \overline{T}(v + \langle v_1\rangle) = Tv + \langle v_1\rangle.$$

By induction there exists a basis $\overline{v}_2, \ldots, \overline{v}_n$ for $V/\langle v_1\rangle$ so that

$$\overline{T}\overline{v}_2 = \alpha_2 \overline{v}_2 \qquad \text{and } \overline{T}\overline{v}_j = \alpha_j \overline{v}_j + \text{ lower terms.}$$

The first equation precisely means

$$Tv_2 = \alpha_2 v_2 + \text{ something times } v_1.$$

Repeating this argument, we are done.

Alternatively, we write $U^{-1}AU$ as

$$\begin{bmatrix} \alpha_1 & * \\ 0 & A_1 \end{bmatrix}$$

where A_1 is $(n-1)\times(n-1)$. Then the char poly is just $(x-\alpha_1)\cdot$ char poly of A_1. Repeating this process, conjugating now with

$$\begin{bmatrix} 1 & 0 \\ 0 & u_1 \end{bmatrix}$$

and so on, we obtain the claim. □

Theorem: Cayley-Hamilton

The min polynomial of a matrix divides its char poly.

Proof. We first consider matrices in $\mathbb{Z}[x_{i,j}]$, or equivalently polynomials in n^2 variables over $\mathbb{Z}$. The char poly of X, $\varphi(t)$, is

$$\det(tI - X) = t^n + \rho_1 t^{n-1} + \rho_2 t^{n-2} + \dots + \rho_n$$

where the ρ's are also polynomials over $\mathbb{Z}$. We claim that it suffices to show that $\varphi(X) = 0$.

To see this, let R be any commutative ring and $A \in M_n(R)$. There exists a homomorphism from $\mathbb{Z}[x_{i,j}] \to R$ by $x_{i,j} \mapsto a_{i,j}$ with $1 \mapsto 1$. This gives a homomorphism on polynomials and matrices. In particular, $\varphi(X) \mapsto$ char poly of A evaluated at A, so if $\varphi(X) = 0$, so is any of its homomorphic image, char poly of A included.

Since $\mathbb{Z}[x_{i,j}]$ embeds into $\mathbb{C}$ (i.e., there exists n^2 elements in $\mathbb{C}$ algebraically independent), it suffices to prove the $\mathbb{C}$ case. For complex matrices, however, we can conjugate it into upper triangular and perturb the diagonal entries. The corresponding char poly converges to that of the original upper triangular too, whereas the perturbed matrices converge to the triangular matrix too. Each perturbed char poly has $f_\epsilon(A_\epsilon) = 0$. Taking limits, $f(A) = 0$. □

Theorem

Let $A, B \in M_n(F)$. Assume $AB = BA$.

- If A and B are both diagonalizable, we can find a basis that works for both A and B.
- In general, they can be put into upper triangular forms simultaneously.

Proof. We prove the second claim. Let α_1 be an eigenvalue for A and let V_1 be the α_1-eigenspace.

We claim $BV_1 \subset V_1$: if $Av = \alpha_1 v$, then $A(Bv) = BAv = \alpha_1 Bv$, so Bv is also contained in V_1. Therefore B on V_1 is a linear transformation, and there exists $v \neq v_1 \in V_1$ with $Bv_1 = \beta_1 v_1$. Obviously $Av_1 = \alpha_1 v_1$. The fact that $AB = BA$ implies $A_1 B_1 = B_1 A_1$ in

$$A = \begin{bmatrix} \alpha_1 & * \\ 0 & A_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \beta_1 & * \\ 0 & B_1 \end{bmatrix}.$$

And the induction continues. □

Theorem

Let $A \in M_n(\mathbb{C})$. Then there exists a unitary matrix U with UAU^{-1} is upper triangular.

Proof. Let v_1 be an eigenvector of A. WLOG assume $\|v_1\| = 1$. Let $v_1, \dots, v_n$ be an orthonormal basis and we perform change of basis:

$$\begin{bmatrix} \alpha_1 & * \\ 0 & A_1 \end{bmatrix}.$$

By induction, we find an orthonormal basis for the span of $v_2, \dots, v_n$ so A_1 becomes upper triangular. Repeating the process gives the claim. □

Companion Matrices

Suppose we are given $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$. Let A_f be the **companion matrix** of f defined by

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}.$$

We claim that the char poly of A_f is the min poly of A_f, and they are both f.