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Jordan Canonical Form

Let $A \in M_n(F)$. Let its char poly be $f(x) = (x-\alpha_1)^{e_1} \ldots (x-\alpha_r)^{e_r}$ where the α_i's are distinct and $e_i > 0$.
Case 1. $f(x) = x^e$ so 0 is the only eigenvalue. In this case anything that divides x^e are just x^{e_i}, so the rational canonical form is block diagonal $C_{x^{e_1}}, \ldots, C_{x^{e_r}}$ with $e_1 \geqslant e_2 \geqslant \ldots \geqslant e_r$. In particular, each companion matrix's last column is just zero column. In this case, the Jordan canonical form is just defined to be the rational canonical form.
Case 2. Now assume $f(x) = (x-\alpha)^e$. We make a change of variable and consider $A - \alpha I$, which is similar to block diagonal $C_{x^{e_1}}, \ldots, C_{x^{e_r}}$. If we call the matrix in case 1 N, then $A - \alpha I$ is similar to N, and A is similar to $\alpha I + N$.
We define a **Jordan block** to be an $\ell \times \ell$ matrix $\lambda I + C_{x^\ell}$, namely, the matrix with λ on diagonal entries, 1 on the shorter diagonal under, and 0 everywhere else.

Theorem: Jordan Canonical Form

If the char poly of A is a product of linear factors, then A is similar to a block diagonal matrix $J_{\lambda_1, e_1}, J_{\lambda_2, e_2}, \ldots, J_{\lambda_r, e_r}$.

Corollary

If A is in $M_n(F)$ then A is similar to A^T, with $A^T = S^{-1}AS$ where S is symmetric and invertible.

Proof. **Step 1.** If $A \sim B$ and B is symmetrically similar to B^T then A is symmetrically similar to A^T (i.e., there is a symmetrical invertible matrix converting A to A^T).
To see this, assume $A = U^{-1}BU$ for some U and $B^T = S^{-1}BS$ with S symmetric. Then $A^T = U^T B U^{-T} = \ldots$
Step 2. The previous step shows it suffices to prove the claim on something similar to A. Here we consider Jordan blocks. Let $A = J_{\lambda,e}$. Take $S = I^T$. Then $S^2 = I$, $S^{-1} = S$, and $S^{-1}AS = J_{\lambda,e}^T$.
Step 3. Let A be a Jordan block again. We show that if $U^{-1}AU = A^T$ then U is symmetric. This is because $U^{-1}AU = S^{-1}AS$, so

$$SU^{-1}A(US^{-1}) = A \implies (SU^{-1}) \text{ commutes with } A.$$

Therefore, (SU^{-1}) is a polynomial of A, i.e., $SU^{-1} = f(A)$, so $U^{-1} = f(A)S$. This is clearly symmetric: $(f(A)S)^T = S^T f(A)^T = Sf(A^T) = f(A)S$ because $SA = A^T S$.
Step 4. Cyclic case. If $A \sim C_{f(x)}$, it is similar to A^T over some bigger field via a symmetric matrix S. All similarities are symmetric in this case by step 3. Since A and A^T are similar over a bigger field, they are also similar over the smaller field.
Step 5. General case. If A is block diagonal $C_{f_1}, \ldots, C_{f_r}$, for each i we take S_i with $S_i^T C_{f_i} S_i = C_{f_i}^T$ where S_i is symmetric. Then $S :=$ block diagonal $S_1, \ldots, S_r$ will finish the proof. □